

Some handwritten notes for
“Fisica Teorica 2 / Advanced QFT”

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SECOND QUANTIZATION

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For non interacting particles $H = \sum_{\text{part.}} H_{sp}$, with H_{sp} the single particle Hamiltonian, e.g. $H = \sum_i \left(\frac{1}{2m} P_i^2 + V(x_i) \right)$ where i labels the particles. In the following we assume the particles to be identical, and we denote by $|m\rangle$ the single particle states, solution of $H_{sp}|m\rangle = E_m|m\rangle$, and by $\psi_m(x) = \langle x|m\rangle$ the corresponding wave-function.

A generic N particle state is written as follows

$$|\lambda_1, \dots, \lambda_N\rangle = \frac{1}{\sqrt{N!} \prod_j \sqrt{m_j!}} \sum_{\text{perm}} \Delta_{\sigma_1 \dots \sigma_N} |\lambda_{\sigma_1}\rangle \dots |\lambda_{\sigma_N}\rangle$$

$\uparrow$ $\uparrow$ $\uparrow$
 $\#$ particles in the j -th state $\quad \quad \quad \Delta_{\sigma_1 \dots \sigma_N} = \begin{cases} 1 & \text{bosons} \\ (-1)^{\text{parity of perm}} & \text{fermions} \end{cases}$

where $\sigma_1, \dots, \sigma_N$ is a permutation of $1, \dots, N$ and obviously $m_j = 0, 1$ for fermions.

In coordinate space

$$\langle x_1, \dots, x_N | \lambda_1, \dots, \lambda_N \rangle = \psi_{\lambda_1 \dots \lambda_N}(x_1, \dots, x_N) = \frac{1}{\sqrt{N!} \prod_j \sqrt{m_j!}} \sum_{\text{perm}} \Delta_{\sigma_1 \dots \sigma_N} \psi_{\lambda_{\sigma_1}}(x_1) \dots \psi_{\lambda_{\sigma_N}}(x_N)$$

and in the fermionic case we can write $\psi_{\lambda_1 \dots \lambda_N}(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det_{ij} [\psi_{\lambda_i}(x_j)]$

(Slater determinant)

For fermions $m_j = 1$, all states $\lambda_1, \dots, \lambda_N$ are different and the result easily follows. For bosons

$$\begin{aligned} \langle \lambda_1, \dots, \lambda_N | \lambda_1, \dots, \lambda_N \rangle &= \frac{1}{N! \prod_j m_j!} \left(\sum_{\text{perm}'} \langle \lambda_{\sigma'_1} | \dots \langle \lambda_{\sigma'_N} | \right) \left(\sum_{\text{perm}} |\lambda_{\sigma_1}\rangle \dots |\lambda_{\sigma_N}\rangle \right) \\ &= \frac{1}{\prod_j m_j!} \langle \lambda_1 | \dots \langle \lambda_N | \left(\sum_{\text{perm}} |\lambda_{\sigma_1}\rangle \dots |\lambda_{\sigma_N}\rangle \right) = 1 \end{aligned}$$

All the $N!$ perm. $\sigma'_1 \dots \sigma'_N$ have the same nonvanishing matrix elements

$\uparrow$
only the $\prod_j m_j!$ permutations with the m_j states cycled have matrix element $\neq 0$

Compact notation for $|\lambda_1, \dots, \lambda_N\rangle$: $|m_1, m_2, \dots\rangle$, where $m_i = \#$ particles in the state i . For bosons $m_i \in \mathbb{N}$, while for fermions $m_i = 0, 1$. A generic N particle state can thus be written

$|\psi\rangle = \sum_{\substack{m_1, m_2, \dots \\ \sum m_i = N}} c_{m_1, m_2, \dots} |m_1, m_2, \dots\rangle$ and these states generate $F^{(N)}$, the Fock space with fixed number of particles. The generic Fock space is $F = \bigoplus_{n=0}^{\infty} F^{(n)}$, and the generic state

$$|\psi\rangle = \sum_{m_1, m_2, \dots} c_{m_1, m_2, \dots} |m_1, m_2, \dots\rangle \quad \text{with} \quad \sum_{m_1, m_2, \dots} |c_{m_1, m_2, \dots}|^2 = 1$$

Creation and annihilation operators are defined on the base $|m_1, m_2, \dots\rangle$ by

$$\begin{aligned} a_k^\dagger |m_1, \dots, m_k, \dots\rangle &= \sqrt{m_k+1} |m_1, \dots, m_k+1, \dots\rangle \\ a_k |m_1, \dots, m_k, \dots\rangle &= \sqrt{m_k} |m_1, \dots, m_k-1, \dots\rangle \end{aligned}$$

$S = +1/-1$ for bosons/fermions
 $S_k = \sum_{j < k} m_j$

and for fermions $m+1$ is to be interpreted mod 2

i.e. $2=0$.

$$\langle m'_1, \dots, m'_k, \dots | a_k^\dagger | m_1, \dots, m_k, \dots \rangle = \sqrt{m_k+1} \sum S_k \langle m'_1, \dots, m'_k, \dots | m_1, \dots, m_{k+1}, \dots \rangle$$

$\neq 0$ only if $m'_j = m_j$ when $k \neq j$ and $m'_k = m_{k+1}$, moreover

$$\langle m'_1, \dots, m'_k, \dots | a_k^\dagger | m_1, \dots, m_k, \dots \rangle = \langle m_1, \dots, m_k, \dots | a_k | m'_1, \dots, m'_k, \dots \rangle^* \quad \text{hence (note that } S_k = S'_k)$$

$a_k | m'_1, \dots, m'_k, \dots \rangle = \sqrt{m'_k} e^{S'_k} | m'_1, \dots, m'_{k-1}, \dots \rangle$ and a_k is thus the hermitian conjugate of the operator $a_k^\dagger$

It is simple to verify that: $a_j a_k^\dagger - \sum a_k^\dagger a_j = \delta_{kj}$, $a_j a_k - \sum a_k a_j = 0$, $a_j^\dagger a_k^\dagger - \sum a_k^\dagger a_j^\dagger = 0$

If $j < k$ we have

$$a_j a_k^\dagger | m_1, \dots, m_j, \dots, m_k, \dots \rangle = \sqrt{m_k+1} \sum S_k a_j | m_1, \dots, m_j, \dots, m_{k+1}, \dots \rangle = \overset{m_{k+1} \text{ has no effect on } S_j \text{ since } j < k}{=} \\ = \sqrt{m_k+1} \sqrt{m_j} \sum S_k + S_j | m_1, \dots, m_{j-1}, \dots, m_{k+1}, \dots \rangle$$

$$a_k^\dagger a_j | m_1, \dots, m_j, \dots, m_k, \dots \rangle = \sqrt{m_j} \sum S_j a_k^\dagger | m_1, \dots, m_{j-1}, \dots, m_k, \dots \rangle = \\ = \sqrt{m_j} \sqrt{m_k+1} \sum S_k + S_j - 1 | m_1, \dots, m_{j-1}, \dots, m_{k+1}, \dots \rangle$$

and if $j > k$ we can follow the same strategy. Hence $a_j a_k^\dagger - \sum a_k^\dagger a_j = 0$ if $j \neq k$ and in the same way $a_j a_k - \sum a_k a_j = 0$ and $a_j^\dagger a_k^\dagger - \sum a_k^\dagger a_j^\dagger = 0$ for any j, k .

If $j = k$ we have

$$\begin{aligned} \text{.) fermions} \quad a_j^\dagger a_j | m_1, \dots, m_j, \dots \rangle &= \begin{cases} m_j=0 & 0 \\ m_j=1 & \sum S_j a_j^\dagger | m_1, \dots, \overbrace{m_{j-1}}^{=0}, \dots \rangle = | m_1, \dots, m_j, \dots \rangle \end{cases} \\ a_j a_j^\dagger | m_1, \dots, m_j, \dots \rangle &= \begin{cases} m_j=1 & 0 \\ m_j=0 & \sum S_j a_j | m_1, \dots, \overbrace{m_{j+1}}^{=1}, \dots \rangle = | m_1, \dots, m_j, \dots \rangle \end{cases} \end{aligned}$$

$$\text{.) bosons} \quad a_j^\dagger a_j | m_1, \dots, m_j, \dots \rangle = \sqrt{m_j} a_j^\dagger | m_1, \dots, m_{j-1}, \dots \rangle = m_j | m_1, \dots, m_j, \dots \rangle$$

$$a_j a_j^\dagger | m_1, \dots, m_j, \dots \rangle = \sqrt{m_j+1} a_j | m_1, \dots, m_{j+1}, \dots \rangle = (m_j+1) | m_1, \dots, m_j, \dots \rangle$$

The state $|m_1, \dots\rangle$ can now be written $|m_1, \dots\rangle = \frac{(a_1^\dagger)^{m_1}}{\sqrt{m_1!}} \frac{(a_2^\dagger)^{m_2}}{\sqrt{m_2!}} \dots |0, 0, \dots\rangle$

We have defined creation and annihilation operators using a specific base, and on the single particle states we have $|k\rangle = a_k |0\rangle$. If we want to change the single particle basis from $|k\rangle$ to $|\lambda\rangle$ we use

$$a_\lambda^\dagger |0\rangle = |\lambda\rangle = \sum_k |k\rangle \langle k|\lambda\rangle = \sum_k \langle k|\lambda\rangle a_k^\dagger |0\rangle$$

to identify the transformation $a_\lambda^\dagger = \sum_k a_k^\dagger \langle k|\lambda\rangle$ (and thus $a_\lambda = \sum_k \langle \lambda|k\rangle a_k$)

In particular, passing to momentum to coordinate space we have $a_x = \sum_k \langle x|k\rangle a_k$

and on a finite volume $\langle x|k\rangle = \frac{1}{\sqrt{V}} e^{ikx}$, while for $V \rightarrow \infty$ $\langle x|k\rangle = \frac{1}{(2\pi)^{D/2}} e^{ikx}$

Note that sometimes a_x is indicated by ψ_x and called field operator (see, e.g., FETTER, WALECKA "Quantum theory of many-particle systems" § I.2)

Let us now consider a generic single particle operator $O_1 = \sum_{i=1}^N V_{(i)}$ with $V_{(i)}$ an operator acting on the i -th particle. On single particle states we have (no sum on i implied)

$$V_{(i)} |\alpha_i\rangle = \sum_{\beta} V_{\beta\alpha} |\beta_i\rangle \quad \text{hence} \quad V_{\beta\alpha} = \langle \beta_i | V_{(i)} | \alpha_i \rangle \quad \text{equal for all } i$$

The action of O_1 on $|\alpha_1, \dots, \alpha_N\rangle$ (where α_i is the state of the i -th particle) is

$$O_1 |\alpha_1, \dots, \alpha_i, \dots, \alpha_N\rangle = \sum_{\beta} \sum_{i=1}^N V_{\beta\alpha_i} |\alpha_1, \dots, \alpha_i \rightarrow \beta, \dots, \alpha_N\rangle$$

where $|\alpha_1, \dots, \alpha_i \rightarrow \beta, \dots, \alpha_N\rangle$ denotes the state in which the i -th particle passed from the state α_i to β . It is possible to see that $|\alpha_1, \dots, \alpha_i \rightarrow \beta, \dots, \alpha_N\rangle = a_{\beta}^{\dagger} a_{\alpha_i} |\alpha_1, \dots, \alpha_i, \dots, \alpha_N\rangle$

$$\text{hence } O_1 = \sum_{\beta\alpha} V_{\beta\alpha} a_{\beta}^{\dagger} a_{\alpha}$$

In the fermionic case all the α_i s have to be different from each other, and

$$\begin{aligned} a_{\beta}^{\dagger} a_{\alpha_i} |\alpha_1, \dots, \alpha_i, \dots, \alpha_N\rangle &= a_{\beta}^{\dagger} a_{\alpha_i} a_{\alpha_1}^{\dagger} \dots a_{\alpha_i}^{\dagger} \dots a_{\alpha_N}^{\dagger} |0\rangle = \overset{a_{\beta}^{\dagger} a_{\alpha_i} \text{ commutes with } a_{\alpha_j}^{\dagger} \text{ with } j < i}{=} \\ &= a_{\alpha_1}^{\dagger} \dots a_{\beta}^{\dagger} a_{\alpha_i} a_{\alpha_i}^{\dagger} \dots a_{\alpha_N}^{\dagger} |0\rangle = \overset{a_{\alpha_i} a_{\alpha_i}^{\dagger} + a_{\alpha_i}^{\dagger} a_{\alpha_i} = 1}{=} \\ &= a_{\alpha_1}^{\dagger} \dots a_{\beta}^{\dagger} (1 - a_{\alpha_i}^{\dagger} a_{\alpha_i}) \dots a_{\alpha_N}^{\dagger} |0\rangle = \overset{a_{\alpha_i} \text{ anticommutes with } a_{\alpha_j}^{\dagger} \text{ with } j > i \text{ and } a_{\alpha_i} |0\rangle = 0}{=} \\ &= a_{\alpha_1}^{\dagger} \dots a_{\beta}^{\dagger} \dots a_{\alpha_N}^{\dagger} |0\rangle = |\alpha_1, \dots, \alpha_i \rightarrow \beta, \dots, \alpha_N\rangle \end{aligned}$$

As a consequence $O_1 = \sum_{\beta} \sum_{i=1}^N V_{\beta\alpha_i} a_{\beta}^{\dagger} a_{\alpha_i}$, but the sum on i can be written as a sum on any α , since a_{α} gives zero whenever the state α is not occupied, hence $O_1 = \sum_{\beta\alpha} V_{\beta\alpha} a_{\beta}^{\dagger} a_{\alpha}$.

In the bosonic case we can write

$$\sum_{i=1}^N V_{\beta\alpha_i} |\alpha_1, \dots, \alpha_i \rightarrow \beta, \dots, \alpha_N\rangle = \sum_k V_{\beta k} \sqrt{m_k} \sqrt{m_{\beta}+1} |\dots, m_k-1, \dots, m_{\beta}+1, \dots\rangle$$

where $\sqrt{m_k}$ comes from the simplification of m_k in the sum and $\frac{1}{\sqrt{m_k}}$ in the normalization of the state, while $\sqrt{m_{\beta}+1}$ appears since in the original normalization a factor $\frac{1}{\sqrt{m_{\beta}}!}$ was present, which becomes $1/\sqrt{(m_{\beta}+1)!}$. The coefficient $\sqrt{m_k} \sqrt{m_{\beta}+1}$ are exactly those produced by $a_{\beta}^{\dagger} a_k$.

Operators which are diagonal in the single particle basis can thus be written as

$$O_1 = \sum_{\alpha} D_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} = \sum_{\alpha} D_{\alpha} N_{\alpha}$$

where N_{α} is the number operator for the state α . Example: in the momentum eigenstates basis $E_{kin} = \sum_i \frac{1}{2m} p_i^2 = \sum_k \frac{1}{2m} \hbar^2 k^2 a_k^{\dagger} a_k$

Let us now consider two particles operators: $O_2 = \frac{1}{2} \sum_{i \neq j} V_{(ij)}$. Note that the constraint $i \neq j$ excludes self-interactions, which are single particle processes. In the single particle basis we assume (no sum on i, j)

$$V_{(ij)} |\alpha_j\rangle |\beta_i\rangle = \sum_{\gamma\delta} V_{\gamma\delta\alpha\beta} |\gamma_i\rangle |\delta_j\rangle \quad (\text{attention to } i, j \text{ order})$$

hence $V_{\gamma\delta\alpha\beta} = \langle \gamma_i | \langle \delta_j | V_{(ij)} | \alpha_j \rangle | \beta_i \rangle = \int \psi_\gamma^*(x_i) \psi_\delta^*(x_j) V(x_i, x_j) \psi_\alpha(x_j) \psi_\beta(x_i) dx_i dx_j$

and we have, in the single particle basis

$$O_2 | \alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_N \rangle = \frac{1}{2} \sum_{\gamma\delta} \sum_{i \neq j} V_{\gamma\delta\alpha_j\alpha_i} | \alpha_1, \dots, \alpha_j \rightarrow \delta, \dots, \alpha_i \rightarrow \gamma, \dots, \alpha_N \rangle$$

which can be rewritten as

$$O_2 = \frac{1}{2} \sum_{\gamma\delta\alpha\beta} V_{\gamma\delta\alpha\beta} a_\gamma^\dagger a_\delta^\dagger a_\alpha a_\beta$$

For the fermionic case it is simple to verify that

$$| \alpha_1, \dots, \alpha_j \rightarrow \delta, \dots, \alpha_i \rightarrow \gamma, \dots, \alpha_N \rangle = a_\gamma^\dagger a_\delta^\dagger a_{\alpha_j} a_{\alpha_i} | \alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_N \rangle$$

E.g.: if $\gamma, \delta, \alpha_j, \alpha_i$ are all different we have $a_\gamma^\dagger a_\delta^\dagger a_{\alpha_j} a_{\alpha_i} = (a_\gamma^\dagger a_{\alpha_i})(a_\delta^\dagger a_{\alpha_j})$ and we can use the result found before for the single body operator. If $\gamma = \alpha_j \neq \delta = \alpha_i$ then

$$| \alpha_1, \dots, \alpha_j \rightarrow \delta, \dots, \alpha_i \rightarrow \gamma, \dots, \alpha_N \rangle = | \alpha_1, \dots, \alpha_i, \dots, \alpha_j, \dots, \alpha_N \rangle = - | \alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_N \rangle$$

$$\text{and } a_\gamma^\dagger a_\delta^\dagger a_{\alpha_j} a_{\alpha_i} = a_{\alpha_j}^\dagger a_{\alpha_i}^\dagger a_{\alpha_j} a_{\alpha_i} = - a_{\alpha_j}^\dagger a_{\alpha_j} a_{\alpha_i}^\dagger a_{\alpha_i} \quad \text{hence}$$

$$a_\gamma^\dagger a_\delta^\dagger a_{\alpha_j} a_{\alpha_i} | \alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_N \rangle = - a_{\alpha_j}^\dagger a_{\alpha_j} a_{\alpha_i}^\dagger a_{\alpha_i} | \alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_N \rangle$$

ok

The bosonic case requires more care to verify that all $\sqrt{n}$ are where they should. For example,

if $\alpha_i = \alpha_j$ and there are n particles (among which the i and j) in the same state, from

$$\sum_{\gamma\delta} \sum_{i \neq j} V_{\gamma\delta\alpha_j\alpha_i} | \alpha_1, \dots, \alpha_j \rightarrow \delta, \dots, \alpha_i \rightarrow \gamma, \dots, \alpha_N \rangle \quad \text{we get } n(n-1) \text{ equal contributions for each } \gamma, \delta.$$

If $\gamma \neq \alpha_j$, $\delta \neq \alpha_j$ and $\gamma \neq \delta$ we have, considering normalizations

$$\sum_{i \neq j} | \alpha_1, \dots, \alpha_j \rightarrow \delta, \dots, \alpha_i \rightarrow \gamma, \dots, \alpha_N \rangle = n(n-1) \frac{\sqrt{m_{\delta+1}}}{\sqrt{m_{\gamma+1}}} \frac{1}{\sqrt{m(m-1)}} | \dots, m_{\delta+1}, \dots, m_{\gamma+1}, \dots, m-2 \rangle$$

$$\text{and } a_\gamma^\dagger a_\delta^\dagger a_{\alpha_j} a_{\alpha_i} | \alpha_1, \dots, \alpha_j, \dots, \alpha_i, \dots, \alpha_N \rangle = \frac{\sqrt{m_{\delta+1}}}{\sqrt{m_{\gamma+1}}} \frac{1}{\sqrt{m(m-1)}} | \dots, m_{\delta+1}, \dots, m_{\gamma+1}, \dots, m-2 \rangle$$

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 $\alpha_j = \alpha_i$

(for explicit computations in one- and two-body operators we approximately followed

GIULIANI, VIGNALE "Quantum theory of the electron liquid")

SINGLE SPIN PATH-INTEGRAL

$\hbar=1$

$H = -J\sigma_1 - h\sigma_3$ with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

when not stated otherwise we will use the basis $|s\rangle$ in which σ_3 is diagonal: $\sigma_3|s\rangle = s|s\rangle$.

We are interested in computing $\langle s_f | e^{-iH(t_f-t_i)} | s_i \rangle =$ probability amplitude of observing the eigenvalue s_f of σ_3 at time t_f if the system was prepared in s_i at time t_i .

$e^{-iH(t_f-t_i)} = [e^{-iH(t_f-t_i)/N}]^N$ and we introduce $a = \frac{t_f-t_i}{N}$

$\langle s_f | e^{-iH(t_f-t_i)} | s_i \rangle = \langle s_f | e^{-iaH} e^{-iaH} \dots e^{-iaH} | s_i \rangle$

we insert $1 = \sum_{s_j=\pm 1} |s_j\rangle \langle s_j|$, using "j" to remember the time at which 1 has been inserted.

$$\begin{aligned} \langle s_f | e^{-iH(t_f-t_i)} | s_i \rangle &= \sum_{\substack{s_j=\pm 1 \\ j=1, \dots, N-1}} \langle s_f | e^{-iaH} | s_{N-1} \rangle \langle s_{N-1} | \dots | s_1 \rangle \langle s_1 | e^{-iaH} | s_i \rangle = \\ &= \sum_{\substack{s_j=\pm 1 \\ j=1, \dots, N-1}} \prod_{j=0}^{N-1} \langle s_{j+1} | e^{-iaH} | s_j \rangle \quad \text{with } s_0 = s_i, s_N = s_f \end{aligned}$$

By writing $\langle s' | e^{-iaH} | s \rangle = \exp\{-iaA(s', s)\}$ we have

$$\langle s_f | e^{-iH(t_f-t_i)} | s_i \rangle = \sum_{\{s_j\}} \exp\left\{-ia \sum_{j=0}^{N-1} A(s_{j+1}, s_j)\right\}$$

and the sum is on all single spin configurations at different times: discrete path-integral, also known as SUZUKI-TROTTER expansion.

Let us now consider on analytic continuation to imaginary time: $t \rightarrow -i\tau$. We have

$\langle s_f | e^{-iH(t_f-t_i)} | s_i \rangle \rightarrow \langle s_f | e^{-\tau H} | s_i \rangle$ and thus, as before ($a = \tau/N$)

$$\langle s_f | e^{-\tau H} | s_i \rangle = \sum_{\{s_j\}} \prod_{j=0}^{N-1} \langle s_{j+1} | e^{-aH} | s_j \rangle \quad \text{always with } s_0 = s_i \text{ and } s_N = s_f$$

Using the CAMPBELL-BAKER-HAUSDORFF formula at the lowest order, $e^{a\sigma_1} e^{a\sigma_3} = e^{a(\sigma_1+\sigma_3)} (1 + O(a^2))$, we get $\langle s' | e^{aJ\sigma_1 + ah\sigma_3} | s \rangle \simeq \langle s' | e^{aJ\sigma_1} e^{ah\sigma_3} | s \rangle = \langle s' | e^{aJ\sigma_1} | s \rangle e^{ahs}$.

$$\langle s' | e^{aJ\sigma_1} | s \rangle = \sum_{s_x=\pm 1} \langle s' | e^{aJ\sigma_1} | s_x \rangle \langle s_x | s \rangle = \sum_{s_x} \langle s' | s_x \rangle \langle s_x | s \rangle e^{aJs_x}$$

$|s_x\rangle$ eigenstates of σ_1 : $|s_x^{(+)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with eigenvalue +1
 $|s_x^{(-)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, with eigenvalue -1

$|s_x^{(+)}\rangle \langle s_x^{(+)}| = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $|s_x^{(-)}\rangle \langle s_x^{(-)}| = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

and using these formulas we get

$$\begin{aligned} \langle s' | e^{aJ\sigma_x} | s \rangle &= \sum_{s_x} \langle s' | s_x \rangle \langle s_x | s \rangle e^{aJ s_x} = \frac{1}{2} e^{aJ} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{s's} + \frac{1}{2} e^{-aJ} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}_{s's} = \\ &= \begin{pmatrix} \cosh(aJ) & \sinh(aJ) \\ \sinh(aJ) & \cosh(aJ) \end{pmatrix}_{s's} = \cosh(aJ) \bar{s}_{s's} + \sinh(aJ) (1 - \bar{s}_{s's}) \end{aligned}$$

Since $s, s' = \pm 1$, we can write $\bar{s}_{ss'} = \frac{1+ss'}{2}$, hence

$$\begin{aligned} \langle s' | e^{aJ\sigma_x} | s \rangle &= \frac{1}{2} \cosh(aJ) + \frac{1}{2} ss' \cosh(aJ) + \frac{1}{2} \sinh(aJ) - \frac{1}{2} ss' \sinh(aJ) = \\ &= \frac{1}{2} e^{aJ} + \frac{1}{2} e^{-aJ} ss' \end{aligned}$$

We now want to write $\langle s' | e^{aJ\sigma_x + ah\sigma_z} | s \rangle = e^{-aA_E(s',s)}$, hence we use

$$\begin{aligned} \frac{1}{2} e^{aJ} + \frac{1}{2} e^{-aJ} ss' &= \exp \left\{ \log \left[\frac{1}{2} e^{aJ} + \frac{1}{2} e^{-aJ} ss' \right] \right\} = \\ &= \exp \left\{ \log(\cosh(aJ)) \bar{s}_{ss'} + \log(\sinh(aJ)) (1 - \bar{s}_{ss'}) \right\} = \\ &= \exp \left\{ a(C_0 + k ss') \right\} \quad \text{with} \quad \begin{cases} C_0 = \frac{1}{2a} \log(\cosh(aJ) \sinh(aJ)) \\ k = \frac{1}{2a} \log(\coth(aJ)) \end{cases} \end{aligned}$$

NOTE: $C_0, k \in \mathbb{R}$ only if $J > 0$

Hence finally

$$\langle s_f | e^{-\tau H} | s_i \rangle \underset{\substack{\text{upto } O(a) \\ s_0 = s_i, s_N = s_f}}{\simeq} \sum_{\{s_j\}, j=1, \dots, N-1} e^{-H_C}$$

with $H_C = -a \sum_{j=0}^{N-1} (C_0 + k s_j s_{j+1} + h s_j)$

↑
classical Hamiltonian of a 1-dim system
with ± 1 variables (Ising model) and fixed b.c.

Partition function: the partition function of the quantum model with $H = -J\sigma_x - h\sigma_z$ is given by $Z = \text{tr}(e^{-\beta H}) = \sum_s \langle s | e^{-\beta H} | s \rangle \underset{\substack{\text{upto } O(a) \\ \text{with } a = \beta/N}}{\simeq} \sum_{\substack{\{s_j\}, j=0, \dots, N-1 \\ s_0 = s_N}} e^{-H_C} = \text{classical partition function of Ising 1d with periodic b.c.}$

The partition function of the quantum model is equal to the partition function of a classical model of N variables, in the limit $N \rightarrow \infty$ with $\beta = aN$ fixed. Basic example of the so-called quantum to classical mapping

We can also consider two point (in fact n-point) functions of the operator σ_z in Heisenberg representation. $\sigma_z(t) = e^{iHt} \sigma_z e^{-iHt}$ and we define the T-ordered product

$$T(\sigma_z(t_1) \sigma_z(t_2)) = \begin{cases} \sigma_z(t_2) \sigma_z(t_1) & \text{if } t_1 < t_2 \\ \sigma_z(t_1) \sigma_z(t_2) & \text{if } t_2 < t_1 \end{cases}$$

If $t_1 < t_2$, considering $t \rightarrow -i\tau$, hence $\sigma_z(\tau) = e^{\tau H} \sigma_z e^{-\tau H}$, we have

$$\begin{aligned}
 t_2 (e^{-\beta H} \sigma_3(t_2) \sigma_3(t_1)) &= t_2 (e^{-\beta H} e^{\tau_2 H} \sigma_3 e^{-\tau_2 H} e^{\tau_1 H} \sigma_3 e^{-\tau_1 H}) = \\
 &= t_2 (e^{-H(\beta-\tau_2)} \sigma_3 e^{-(\tau_2-\tau_1)H} \sigma_3 e^{-\tau_1 H}) \quad \text{inserting } 1 = \sum |s_j\rangle \langle s_j| \text{ as before} \\
 &= \sum_{\substack{\{s_j\}, j=0, \dots, N-1 \\ s_0 = s_N}} S_{\tau_2} S_{\tau_1} e^{-H\tau_1} = \langle S_{\tau_2} S_{\tau_1} \rangle_Z \quad \text{and the same result is found also if } t_2 > t_1
 \end{aligned}$$

In the $T \rightarrow 0$ ($\beta \rightarrow \infty$) limit, $\frac{e^{-\beta H}}{Z} \rightarrow |0\rangle \langle 0|$ and in this limit the expectation value on the fundamental state $\langle 0| T(\sigma_3(t_2) \sigma_3(t_1)) |0\rangle$ is the analytic continuation of the two point function of the Ising model

For a very accurate analysis of the transfer matrix of Ising 2d (classical), which brings to the solution of the model, see

BAXTER "Exactly solved models in statistical physics" § 7

PATH-INTEGRAL IN 1dim QM

$$\begin{aligned}
 \psi(x', t') &= \langle x' | \psi(t') \rangle = \langle x' | e^{-iH(t'-t)} | \psi(t) \rangle = \int dx \langle x' | e^{-iH(t'-t)} | x \rangle \psi(x, t) \\
 \langle x' | e^{-iH(t'-t)} | x \rangle &= \sum_m \langle x' | m \rangle \langle m | x \rangle e^{-iE_m(t'-t)} = \sum_m \psi_m(x') \psi_m^*(x) e^{-iE_m(t'-t)}
 \end{aligned}$$

Let us now rewrite $\langle x_f | e^{-iHt} | x_i \rangle$ using the path-integral formulation, starting from $e^{-iHt} = (e^{-i\alpha H})^N$ with $\alpha = t/N$. We assume $H = T(p) + V(x)$ and use the Campbell-Baker-Hausdorff formula at lowest order, $e^{-i\alpha H} = e^{-i\alpha T} e^{-i\alpha V} (1 + O(\alpha^2))$, to write $\langle x_f | e^{-iHt} | x_i \rangle = \langle x_f | e^{-i\alpha T} e^{-i\alpha V} e^{-i\alpha T} e^{-i\alpha V} \dots e^{-i\alpha T} e^{-i\alpha V} | x_i \rangle$. We insert $1 = \int dx_j dp_j |x_j\rangle \langle x_j| |p_j\rangle \langle p_j|$ where $j=1, \dots, N-1$ label the time of the insertion.

We get a product of terms of the form

$$\begin{aligned}
 &\int dx_{j+1} dp_{j+1} \underbrace{\langle x_{j+1} | p_{j+1} \rangle}_{\frac{1}{\sqrt{2\pi}} e^{i p_{j+1} x_{j+1}}} \underbrace{\langle p_{j+1} | e^{-i\alpha T} e^{-i\alpha V} | x_j \rangle}_{e^{-i\alpha T(p_{j+1})} \langle p_{j+1} | e^{-i\alpha V(x_j)} | x_j \rangle} \\
 &= \int dx_{j+1} \frac{dp_{j+1}}{2\pi} \exp \left\{ i p_{j+1} (x_{j+1} - x_j) - i\alpha T(p_{j+1}) - i\alpha V(x_j) \right\} \quad \text{hence}
 \end{aligned}$$

$$\begin{aligned}
 \langle x_f | e^{-iHt} | x_i \rangle &\underset{\text{up to } O(\alpha)}{\sim} \int \prod_{j=1}^{N-1} \left(dx_j \frac{dp_j}{2\pi} \right) \exp \left\{ -i\alpha \left[\sum_{j=0}^{N-1} V(x_j) + \sum_{j=0}^{N-1} T(p_{j+1}) - \sum_{j=0}^{N-1} \frac{1}{\alpha} p_{j+1} (x_{j+1} - x_j) \right] \right\} \\
 &\quad \text{with } x_0 = x_i, x_N = x_f \quad \text{trajectories with } |p \Delta x| \gg \alpha \text{ are suppressed by this term}
 \end{aligned}$$

Continuum limit: $N \rightarrow \infty$, $a \rightarrow 0$ with $t = aN$ fixed. In this limit

$$a \sum_{j=0}^{N-1} \rightarrow \int_0^t dt \quad \text{and} \quad \frac{x_{j+1} - x_j}{a} \rightarrow \partial_t x \quad (\text{sort of, some more care would be required})$$

$$t_j = a_j \quad V(x_j) + T(p_{j+1}) \rightarrow V(x(t_j)) + T(p(t_j))$$

and we formally define $\lim_{N \rightarrow \infty} \int \prod_{j=1}^N \left(dx_j \frac{dp_j}{2\pi} \right) = \int_{x(0)=x_i}^{x(t)=x_f} Dx Dp$, hence we have

$$\langle x_f | e^{-i\hbar H} | x_i \rangle = \int_{x(0)=x_i, x(t)=x_f} Dx Dp e^{i \int_0^t dt (p \dot{x} - H)}$$

If $T(p) = \frac{p^2}{2m}$ we can explicitly perform the integral in p , which is the functional analogue of the FRESNEL integral, an analytic continuation of the Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$

We thus have

$$\int Dp e^{-i \int_0^t dt \left(\frac{p^2}{2m} - p \dot{x} \right)} = \left(\text{formally divergent factor} \right) e^{\int_0^t dt \frac{(i\dot{x})^2}{4i/(2m)}} = \left(\right) \exp \left\{ i \int_0^t dt \frac{m}{2} \dot{x}^2 \right\}$$

and the formally divergent factor can be reabsorbed in the definition of Dx , which becomes

$$Dx = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i a} \right)^{N/2} \prod_{j=1}^{N-1} dx_j \quad \text{or} \quad \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i a \hbar} \right)^{N/2} \prod_{j=1}^{N-1} dx_j \quad \text{if we use } \hbar.$$

We thus finally have (inserting back $\hbar$)

$$\langle x_f | e^{-i\hbar H/\hbar} | x_i \rangle = \int_{x(0)=x_i, x(t)=x_f} Dx \exp \left\{ \frac{i}{\hbar} \int_0^t L(x, \dot{x}) dt \right\} \quad \text{with} \quad L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - V(x).$$

In the $\hbar \rightarrow 0$ limit we can use the stationary phase approximation to see that the dominant contributions come from paths for which the Lagrangian is stationary, i.e. classical trajectories.

Explicitly (see, e.g. BLEISTEIN, HANDELSMAN "Asymptotic expansions of integrals" § 8.4)

$$\int_{\mathbb{R}^N} \exp \left\{ \frac{i}{\hbar} F(x) \right\} d^N x = \sum_j e^{\frac{i}{\hbar} F(x_j)} \frac{1}{\left| \det \frac{\partial^2 F}{\partial x \partial x} \right|_{x_j}^{1/2}} (2\pi \hbar)^{N/2} \exp \left(i \frac{\pi}{4} \underbrace{\text{sgn} \frac{\partial^2 F}{\partial x \partial x} \Big|_{x_j}}_{\# \text{ positive eig.val} - \# \text{ negative eig.val}} \right)$$

Let us now consider the position operator in Heisenberg representation $x_H(t) = e^{i\hbar H t} x e^{-i\hbar H t}$; we

introduce the notation $|x, t\rangle = e^{i\hbar H t} |x\rangle$ for eigenstates of $x_H(t)$: $x_H(t) |x, t\rangle = x |x, t\rangle$

(note: $|x, t\rangle$ is not the evolution at time t of $|x\rangle$). We have immediately

$$\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | e^{-i\hbar H(t_f - t_i)} | x_i \rangle = \int Dx e^{iS} \quad \text{with} \quad x(t_i) = x_i, x(t_f) = x_f, S = \int_{t_i}^{t_f} L dt$$

moreover, if $t_f > t > t_i$, we have

$$\begin{aligned} \langle x_f, t_f | x_H(t) | x_i, t_i \rangle &= \langle x_f | e^{-i\hbar H(t_f - t)} x e^{-i\hbar H(t - t_i)} | x_i \rangle = \\ &= \int dx \langle x_f | e^{-i\hbar H(t_f - t)} | x \rangle \langle x | x e^{-i\hbar H(t - t_i)} | x_i \rangle = \int dx \langle x_f | e^{-i\hbar H(t_f - t)} | x \rangle x \langle x | e^{-i\hbar H(t - t_i)} | x_i \rangle = \\ &= \int Dx x(t) e^{iS} \quad \text{with} \quad x(t_i) = x_i \quad \text{and} \quad x(t_f) = x_f \end{aligned}$$

If we introduce the chronological product by $T(x_H(t_1)x_H(t_2)) = \begin{cases} x_H(t_2)x_H(t_1) & \text{if } t_2 > t_1 \\ x_H(t_1)x_H(t_2) & \text{if } t_1 > t_2 \end{cases}$
 we have analogously (assuming $t_f > t_2 > t_1 > t_i$)

$$\begin{aligned} \langle x_f, t_f | T(x_H(t_2)x_H(t_1)) | x_i, t_i \rangle &= \langle x_f | e^{-iH(t_f-t_2)} \hat{x} e^{-iH(t_2-t_1)} \hat{x} e^{-iH(t_1-t_i)} | x_i \rangle = \\ &= \int dy dz \langle x_f | e^{-iH(t_f-t_2)} | y \rangle y \langle y | e^{-iH(t_2-t_1)} | z \rangle z \langle z | e^{-iH(t_1-t_i)} | x_i \rangle = \\ &= \int \mathcal{D}x \, x(t_2)x(t_1) e^{iS} \quad \text{and it is immediate to see that the same result is} \\ &\quad \begin{matrix} x(t_i)=x_i \\ x(t_f)=x_f \end{matrix} \quad \text{obtained also if } t_f > t_1 > t_2 > t_i. \end{aligned}$$

Analogously we have

$$\langle x_f, t_f | T(x_H(t_1) \dots x_H(t_m)) | x_i, t_i \rangle = \int \mathcal{D}x \, x(t_1) \dots x(t_m) e^{iS} \quad \begin{matrix} x(t_i)=x_i \\ x(t_f)=x_f \end{matrix}$$

Some explicit examples

1) for the free particle we have

$$\begin{aligned} \int \mathcal{D}x \, e^{\frac{i}{\hbar} \int_{t_i}^{t_f} \frac{m}{2} \dot{x}^2 dt} &= \langle x_f | e^{-i \frac{p^2}{2m\hbar} (t_f-t_i)} | x_i \rangle = \sum_n \langle x_f | n \rangle \langle n | x_i \rangle e^{-i E_n (t_f-t_i)/\hbar} = \\ &\quad \begin{matrix} x(t_i)=x_i \\ x(t_f)=x_f \end{matrix} \\ &= \int \frac{dp}{2\pi\hbar} e^{i p(x_f-x_i)/\hbar - i \frac{p^2}{2m\hbar} (t_f-t_i)/\hbar} = \\ &= \sqrt{\frac{m}{2\pi\hbar i(t_f-t_i)}} \exp \left\{ -\frac{m(x_f-x_i)^2}{2i\hbar(t_f-t_i)} \right\} \end{aligned}$$

2) for the harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ we have

$$\langle x_f | e^{-iHt/\hbar} | x_i \rangle = \sqrt{\frac{m\omega}{2\pi\hbar i \sin(\omega t)}} \exp \left\{ \frac{i m \omega}{2\hbar \sin(\omega t)} [(x_f^2 + x_i^2) \cos(\omega t) - 2x_f x_i] \right\}$$

see, e.g., SCHULMAN "Techniques and applications of path-integration" § 6 and COLEMAN "Aspects of symmetry" § 27. APP 1. Using the canonical formalism the previous result can be obtained by using the MEHLER formula for Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^n}{n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1-z^2}} \exp \left\{ \frac{2xy z - (x^2+y^2)z^2}{1-z^2} \right\}$$

see, e.g., BATEMAN "Higher transcendental functions" vol. II, 10.13 (15)

Path-integral in statistical QM

$$\langle O \rangle = \text{Tr} \left(\frac{e^{-\beta H}}{\mathcal{Z}} O \right) = \frac{1}{\mathcal{Z}} \sum_n e^{-\beta E_n} \langle n | O | n \rangle = \frac{1}{\mathcal{Z}} \int dx_a dx_b \langle x_a | e^{-\beta H} | x_b \rangle \langle x_b | O | x_a \rangle$$

hence the Gibbs density matrix in coordinate representation is $\rho_{ab} = \langle x_a | e^{-\beta H} | x_b \rangle$

If we define the imaginary time evolution with the substitution $t \rightarrow -i\tau$ we have

$\exp(-iHt) \rightarrow \exp(-H\tau)$ and by analytical continuation of the expressions obtained before

we get the path-integral expression for the density operator. From

$$\langle x_a | e^{-iHt} | x_b \rangle = \int_{\substack{x(0)=x_a \\ x(t)=x_b}} \mathcal{D}x \exp \left\{ i \int_0^t \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \right] dt \right\} \quad \text{we have } \left(\frac{dx}{dt} \right)^2 \rightarrow - \left(\frac{dx}{d\tau} \right)^2 \text{ hence}$$

$$iS = i \int_0^t \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \right] dt = - \int_0^\beta \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] d\tau = - \int_0^\beta L_E(x, \frac{dx}{d\tau}) d\tau = - S_E$$

$\uparrow$ Euclidean Lagrangian $\uparrow$ Euclidean action.

$$Z = \int dx \langle x | e^{-\beta H} | x \rangle = \int_{x(0)=x(\beta)} \mathcal{D}x e^{-S_E}$$

initial and final states are the same in the path-integral since they are the same in the trace

For the harmonic oscillator we have

$$Z = \sum_{n=0}^{\infty} \exp \left\{ -\beta \omega \left(n + \frac{1}{2} \right) \right\} = e^{-\beta \omega / 2} \frac{1}{1 - e^{-\beta \omega}} = \frac{1}{2 \sinh(\beta \omega / 2)}$$

and using $x_i = x_f = x$ and $t = -i\beta$ in the previous path-integral expression we have

$$\begin{aligned} \sin(\omega t) &= \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \frac{e^{\beta\omega} - e^{-\beta\omega}}{2i} = \frac{1}{i} \sinh(\beta\omega) \\ \cos(\omega t) &= \frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos(\beta\omega) \\ \cos(\beta\omega) - 1 &= \frac{e^{\beta\omega} + e^{-\beta\omega}}{2} - 1 = \frac{1}{2} (e^{\beta\omega} - 1 + e^{-\beta\omega} - 1) = \\ &= e^{\beta\omega/2} \sinh\left(\frac{\beta\omega}{2}\right) - e^{-\beta\omega/2} \sinh\left(\frac{\beta\omega}{2}\right) = 2 \sinh^2\left(\frac{\beta\omega}{2}\right) \end{aligned}$$

hence

$$\langle x | e^{-\beta H} | x \rangle = \sqrt{\frac{m\omega}{2\pi \sinh(\beta\omega)}} \exp \left\{ -\frac{m\omega x^2}{\sinh(\beta\omega)} 2 \sinh^2(\beta\omega/2) \right\}$$

$$Z = \int \langle x | e^{-\beta H} | x \rangle = \sqrt{\frac{m\omega}{2\pi \sinh(\beta\omega)}} \sqrt{\frac{\pi \sinh(\beta\omega)}{2m\omega \sinh^2(\beta\omega/2)}} = \frac{1}{2 \sinh(\beta\omega/2)} \quad \text{as found before}$$

Generating functional of vacuum correlation functions

We have seen before that

$$(x', t' | T(x_H(t_1) \dots x_H(t_n)) | x, t) = \int_{x,t}^{x',t'} \mathcal{D}x \, x(t_1) \dots x(t_n) e^{iS}$$

and in the following the suffix H will be intended for simplicity. If we introduce the notation

$$\underline{(x', t' | x, t)_S} = \int_{x,t}^{x',t'} \mathcal{D}x \, e^{iS_S} \quad \text{with} \quad S_S = \int (L(x, \dot{x}) + \mathcal{J}(t)x(t)) dt$$

where $\mathcal{J}(t)$ is an external classical source, we then have

$$\langle x', t' | T(x(t_1) \dots x(t_m) | x, t) = \frac{1}{i^m} \frac{\delta^m \langle x', t' | x, t \rangle_S}{\delta x(t_1) \dots \delta x(t_m)} \Big|_{J=0}$$

and $\langle x', t' | x, t \rangle_S$ is thus the generating functional of chronological correlation functions computed between the eigenstates of the Heisenberg representation $|x, t\rangle$ and $|x', t'\rangle$.

We are now interested in computing correlation functions on the ground state (on the vacuum in the QFT case). Using $1 = \int dq |q\rangle \langle q| = \int dq |q, t\rangle \langle q, t|$ we have

$$\langle 0 | T[x(t_1) \dots x(t_m)] | 0 \rangle = \int dx dx' \langle 0 | x', t' \rangle \langle x', t' | T[x(t_1) \dots x(t_m)] | x, t \rangle \langle x, t | 0 \rangle$$

Note that the result is independent of t, t' as far as $t' > t_1, \dots, t_m > t$. Moreover

$$\langle x, t | 0 \rangle = \langle x | e^{-iHt} | 0 \rangle = e^{-iE_0 t} \langle x | 0 \rangle = e^{-iE_0 t} \psi_0(x) = \psi_0(x, t) \quad \text{hence}$$

$$\langle 0 | T[x(t_1) \dots x(t_m)] | 0 \rangle = \int dx dx' \psi_0^*(x', t') \langle x', t' | T[x(t_1) \dots x(t_m)] | x, t \rangle \psi_0(x, t)$$

We can thus introduce the generating functional

$$Z[J] = \lim_{\substack{t \rightarrow -\infty \\ t' \rightarrow +\infty}} \int dx dx' \psi_0^*(x', t') \langle x', t' | x, t \rangle_S \psi_0(x, t) \quad \text{in such a way that}$$

$$\langle 0 | T[x(t_1) \dots x(t_m)] | 0 \rangle = \frac{1}{i^m} \frac{\delta^m Z[J]}{\delta x(t_1) \dots \delta x(t_m)} \Big|_{J=0}$$

Note that the limits in $Z[J]$ have the only purpose of ensuring $t' > t_1, \dots, t_m > t$. We also

$$\text{have} \quad Z[0] = \int dx dx' \langle 0 | x', t' \rangle \langle x', t' | x, t \rangle \langle x, t | 0 \rangle = \langle 0 | 0 \rangle = 1$$

We now show that

$$Z[J] = \lim_{\substack{t \rightarrow +i\infty \\ t' \rightarrow -i\infty}} \frac{\langle x', t' | x, t \rangle_S}{\langle x', t' | x, t \rangle}$$

a result that, in various forms, goes under the name of Gell-Mann Low theorem (see GELL-MANN, Low Phys. Rev. 84, 350 (1951), FETTER, WALECKA "Quantum theory of many particle systems" § III.6, WEINBERG "The quantum theory of fields" vol 1, § 6.4)

The proof relies on the adiabatic switching on of the external source for early times and on the adiabatic switching off for late times. In particular, we can assume the existence of two times t_b and t_a such that $t' > t_b > t_a > t$ and $J(t) \neq 0$ only in (t_a, t_b) . In this case

$$\langle x', t' | x, t \rangle_S = \int dx_a dx_b \langle x', t' | x_b, t_b \rangle \langle x_b, t_b | x_a, t_a \rangle_S \langle x_a, t_a | x, t \rangle$$

moreover we have

$$\langle x', t' | x_b, t_b \rangle = \langle x' | e^{-iH(t'-t_b)} | x_b \rangle = \sum_m \langle x' | m \rangle \langle m | x_b \rangle e^{-iE_m(t'-t_b)} = \sum_m \psi_m(x') \psi_m^*(x_b) e^{-iE_m(t'-t_b)}$$

and analogous expressions for $\langle x_a, t_a | x, t \rangle$ and $\langle x', t' | x, t \rangle$. Putting everything together

we can write

$$\frac{(x', t' | x, t)_S}{(x', t' | x, t)} = \frac{\int dx_a dx_b \left(\sum_n \psi_n^*(x) \psi_n^*(x_b) e^{-iE_n(t'-t_b)} \right) (x_b, t_b | x_a, t_a)_S \left(\sum_k \psi_k(x_a) \psi_k^*(x) e^{-iE_k(t_a-t)} \right)}{\sum_j \psi_j^*(x') \psi_j^*(x) e^{-iE_j(t'-t)}}$$

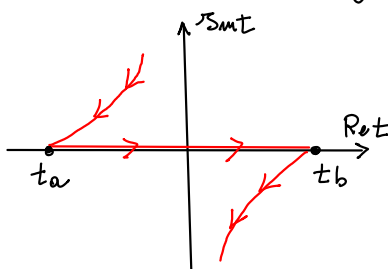
In the sum with $e^{-iE_n(t'-t_b)}$ the term with E_0 (the energy of the fundamental state) dominates in the limit $t' \rightarrow -i\infty$, and the same happens in the limit $t \rightarrow i\infty$ in the sum with $e^{-iE_k(t_a-t)}$. Using only the dominant terms we get

$$\lim_{\substack{t' \rightarrow -i\infty \\ t \rightarrow +i\infty}} \frac{(x', t' | x, t)_S}{(x', t' | x, t)} = \int dx_a dx_b \psi_0^*(x_b) e^{iE_0 t_b} (x_b, t_b | x_a, t_a)_S \psi_0(x_a) e^{-iE_0 t_a} = \\ = \int dx_a dx_b \psi_0^*(x_b, t_b) (x_b, t_b | x_a, t_a)_S \psi_0(x_a, t_a) = Z[S]$$

which complete the proof. Using the path-integral representation we thus have

$$Z[S] = \lim_{\substack{t' \rightarrow -i\infty \\ t \rightarrow +i\infty}} \frac{\int_{x_t}^{x_{t'}} Dx e^{i(S + \int_{t_t}^{t_{t'}} \mathcal{L}(t) x(t) dt)}}{\int_{x_t}^{x_{t'}} Dx e^{iS}}$$

and the integration path in the



complex plane is something like

(note that the details of the path

are irrelevant as far as $\begin{matrix} t' \rightarrow -i\infty \\ t \rightarrow +i\infty \end{matrix}$:

something like



would work as well)

We can now deform the path of integration (without altering the asymptotic conditions at $t' \rightarrow -i\infty$ $t \rightarrow +i\infty$ which project on the fundamental state) to continue the n-points function at imaginary time : $t = -e^{i\theta} \tau$ with $\theta: \pi \rightarrow \pi/2$. In this way we get the generating functional

of Euclidean correlation functions:



$$Z_E[S] = \frac{\int Dx e^{-\int_{-\infty}^{+\infty} (L_E - S(\tau) x(\tau)) d\tau}}{\int Dx e^{-\int_{-\infty}^{+\infty} L_E d\tau}}, \quad \text{Euclidean Lagrangian introduced above}$$

$$L_E = \frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x)$$

$$\langle x(\tau_1) \dots x(\tau_n) \rangle = \frac{\delta^n Z[S]}{\delta S(\tau_1) \dots \delta S(\tau_n)} \Big|_{S=0} = \text{thermal correlation functions at } T=0 \text{ } (\beta=\infty)$$

We thus have $\langle x(\tau_1) \dots x(\tau_n) \rangle \longleftrightarrow \langle 0 | T[x(t_1) \dots x(t_n)] | 0 \rangle$ under $t = -i\tau$.

PATH - INTEGRAL FOR SCALAR FIELD THEORY

conventions: d = space dimension, $D = d+1$ = space-time dimension, signature $(+, \underbrace{-, \dots, -}_d)$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - v(\phi), \quad L = \int \mathcal{L}(\phi, \partial_\mu \phi) d^d x, \quad S = \int L dt = \int d^D x \mathcal{L}$$

$$\mathcal{H} = \frac{1}{2} \pi_\phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + v(\phi), \quad H = \int d^d x \mathcal{H}(\phi, \pi_\phi)$$

3. passing from QM to QFT $\hat{q} \rightarrow \hat{\phi}(x)$. If we discretize the Lagrangian on a lattice along spatial directions we can use spatial coordinates as indices of the dynamical variables. If we denote by "a" the lattice spacing we have ($\mu=0$ denotes the temporal direction)

$$L \simeq \frac{1}{2} \sum_{\vec{x}} a^d (\partial_t \phi_{\vec{x}})^2 - \frac{1}{2} \sum_{\vec{x}, \mu > 0} a^d \frac{(\phi_{\vec{x}+\hat{\mu}a} - \phi_{\vec{x}})^2}{a^2} - \sum_{\vec{x}} a^d v(\phi_{\vec{x}})$$

the corresponding conjugate momenta are $p_{\vec{x}} = \frac{\partial L}{\partial(\partial_t \phi_{\vec{x}})} = a^d \partial_t \phi_{\vec{x}}$, and the Hamiltonian

$$H = \frac{1}{2} \sum_{\vec{x}} \frac{p_{\vec{x}}^2}{a^d} + \frac{1}{2} \sum_{\vec{x}, \mu > 0} a^d \frac{(\phi_{\vec{x}+\hat{\mu}a} - \phi_{\vec{x}})^2}{a^2} + \sum_{\vec{x}} a^d v(\phi_{\vec{x}})$$

Since $[\phi_{\vec{x}}, \phi_{\vec{y}}] = 0$ (commutator at constant time), we can introduce at each position $\vec{x}$ a basis $\{|\phi_{\vec{x}}\rangle\}$, and we get the basis $\prod_{\vec{x}} |\phi_{\vec{x}}\rangle$ for the field state at time $t=0$.

Using this basis we can repeat verbatim all the steps performed in QM, to get (in discretizing the time direction, using Campbell-Baker-Hausdorff formula, we assume for simplicity to use the same lattice spacing already adopted along spatial directions)

$$\langle \phi' | e^{-iH(t'-t)} | \phi \rangle = \int \prod_{\vec{x}, t} d\phi_{\vec{x}}^{(t)} \exp \left\{ i \sum_t a \left[\frac{a^d}{2} \sum_{\vec{x}} \frac{(\phi_{\vec{x}}^{(t+a)} - \phi_{\vec{x}}^{(t)})^2}{a^2} - \frac{a^d}{2} \sum_{\vec{x}, \mu > 0} \frac{(\phi_{\vec{x}+\hat{\mu}a}^{(t)} - \phi_{\vec{x}}^{(t)})^2}{a^2} - a^d \sum_{\vec{x}} v(\phi_{\vec{x}}^{(t)}) \right] \right\}$$

$\phi_{\vec{x}}^{(t)} = \phi(\vec{x})$
 $\phi_{\vec{x}}^{(t')} = \phi'(\vec{x})$

note that if $D = d+1 \geq 2$ these terms do not force the continuity of paths: paths in QFT are much less regular than in QM (first hint of the necessity of renormalization)

and by formally considering

the continuum limit we write

$$\langle \phi' | e^{-iH(t'-t)} | \phi \rangle = \int \mathcal{D}\phi \exp \left\{ i \int_t^{t'} d\tau \int d^d x \mathcal{L}(\phi, \partial_\mu \phi) \right\}$$

$\phi(t, \vec{x}) = \phi(\vec{x})$
 $\phi(t', \vec{x}) = \phi'(\vec{x})$

If we introduce the fields in the Heisenberg representation $\phi_H(t, \vec{x}) = e^{iHt} \phi(\vec{x}) e^{-iHt}$ and the eigenstates $|\phi, t\rangle = e^{iHt} |\phi\rangle$ (thus $\phi_H(t, \vec{x}) |\phi, t\rangle = \phi |\phi, t\rangle$), exactly as in QM we have

$$\langle \phi', t' | T [\phi_H(t_1, \vec{x}_1) \dots \phi_H(t_m, \vec{x}_m)] | \phi, t \rangle = \int_{\phi, t}^{\phi', t'} \mathcal{D}\phi \phi(t_1, \vec{x}_1) \dots \phi(t_m, \vec{x}_m) e^{iS}$$

and in the following the suffix H will be implied: we will always consider fields in the Heisenberg representation.

We denote by $G(x_1, \dots, x_n) = \langle 0 | T[\phi(x_1) \dots \phi(x_n)] | 0 \rangle$ the correlation functions on the vacuum, and we denote by $Z[\mathcal{J}]$ their generating functional, i.e.

$$Z[\mathcal{J}] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^D x_1 \dots d^D x_n G(x_1, \dots, x_n) \mathcal{J}(x_1) \dots \mathcal{J}(x_n); \quad G(x_1, \dots, x_n) = \frac{1}{i^n} \frac{\delta^n Z[\mathcal{J}]}{\delta \mathcal{J}(x_1) \dots \delta \mathcal{J}(x_n)} \Big|_{\mathcal{J}=0}$$

and using the same technique used in QM we can show that

$$Z[\mathcal{J}] = \lim_{\substack{t \rightarrow +i\infty \\ t' \rightarrow -i\infty}} \frac{\int \mathcal{D}\phi \exp \left\{ i \int_{t'}^t (\mathcal{L} + \mathcal{J}(x)\phi(x)) d^D x \right\}}{\int \mathcal{D}\phi \exp \left\{ i \int_{t'}^t \mathcal{L} d^D x \right\}} \quad \text{or, with a continuation to Euclidean time, } t = -i\tau$$

$$Z_E[\mathcal{J}] = \frac{\int \mathcal{D}\phi \exp \left\{ - \int_{-\infty}^{+\infty} (\mathcal{L}_E - \mathcal{J}(x)\phi(x)) d^D x \right\}}{\int \mathcal{D}\phi \exp \left\{ - \int_{-\infty}^{+\infty} \mathcal{L}_E d^D x \right\}} \quad \text{where } \mathcal{L}_E(\phi, \partial_\mu \phi) = \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi)$$

Note that Lorentz invariance $O(1, d)$ becomes Euclidean invariance $O(D)$ after the imaginary time continuation.

Up to now we have only continued the time coordinate from Minkowsky to Euclidean, but we will need soon to continue also energies; the conventions adopted are the following:

$$M: (x^0, \vec{x}) \rightarrow E: (x_4, \vec{x}) \quad \text{with } x^0 = -i x_4$$

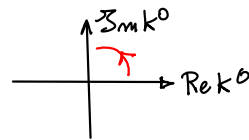
$$\begin{array}{ccc} x_\mu x^\mu = (x^0)^2 - \vec{x}^2 & = & -(x_4)^2 - \vec{x}^2 = -(x_\mu)^2 \\ \uparrow & & \uparrow \\ \text{Lorentzian scal. prod.} & & \text{Euclidean scal. prod.} \end{array}$$



$$M: (k^0, \vec{k}) \rightarrow E: (k_4, \vec{k}) \quad \text{with } k^0 = +i k_4$$

$$k_\mu k^\mu = -(k_\mu)^2$$

$$x_\mu k^\mu = x^0 k^0 - \vec{x} \cdot \vec{k} = x_4 k_4 - \vec{x} \cdot \vec{k}$$



Fourier Transforms keep the same form when passing from Minkowski to Euclidean and vice versa.

FREE FIELD/GAUSSIAN INTEGRALS

Let us start from the Euclidean case: $\mathcal{L}_E = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2$ ($t = -i\tau$). The generating functional is just a functional generalization of a Gaussian integral.

$$1) \text{ 1-dim case } \int_{-\infty}^{+\infty} \exp \left\{ -\frac{1}{2} a x^2 + b x \right\} dx = \sqrt{\frac{2\pi}{a}} \exp \left(\frac{b^2}{2a} \right)$$

$$2) \text{ N-dim case: } A = \text{real symmetric } N \times N \text{ matrix, } A = O^T D O \text{ with } O = \text{orthogonal matrix,}$$

$D = \text{diagonal matrix}$ and we assume all eigenvalues λ_i are strictly positive

It will also be useful to note that A symmetric $\Rightarrow A^{-1}$ symmetric, indeed

$$A A^{-1} = 1 \Rightarrow (A^{-1})^T A^T = 1 \Rightarrow (A^{-1})^T A = 1 \quad \text{and thus } (A^{-1})^T = A^{-1}$$

$$\int d^N x \exp\left\{-\frac{1}{2} x^t A x\right\} = \int d^N y \exp\left\{-\frac{1}{2} y^t B y\right\} = \prod_{j=1}^N \int dy_j \exp\left\{-\frac{\lambda_j}{2} y_j^2\right\} = \prod_{j=1}^N \sqrt{\frac{2\pi}{\lambda_j}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

$y = O x, d^N x = d^N y$ since $O^t O = 1 \Rightarrow |\det O| = 1$

Let us now consider the case in which a source is present

$$\begin{aligned} \int d^N x \exp\left\{-\frac{1}{2} x^t A x + J^t x\right\} &= \int d^N y \exp\left\{-\frac{1}{2} (y + A^{-1} J)^t A (y + A^{-1} J) + J^t (y + A^{-1} J)\right\} = \\ &= \int d^N y \exp\left\{-\frac{1}{2} y^t A y - \cancel{\frac{1}{2} J^t y} - \cancel{\frac{1}{2} y^t J} - \frac{1}{2} J^t A^{-1} J + \cancel{J^t y} + J^t A^{-1} J\right\} = e^{\frac{1}{2} J^t A^{-1} J} \int d^N y e^{-\frac{1}{2} y^t A y} = \\ &= \exp\left(\frac{1}{2} J^t A^{-1} J\right) \frac{(2\pi)^{N/2}}{\sqrt{\det A}} \end{aligned}$$

3) 2-dim case. In analogy with the previous case we have

$$\int D\phi \exp\left\{-\frac{1}{2} \int \phi(x) A(x, y) \phi(y) dx dy + \int b(x) \phi(x) dx\right\} = \frac{N}{\sqrt{\det A}} \exp\left\{\frac{1}{2} \int b(x) A^{-1}(x, y) b(y) dx dy\right\}$$

where $A^{-1}(x, y)$ is defined by $\int A(x, y) A^{-1}(y, z) dy = \delta(x - z)$, N is an irrelevant (formally divergent) proportionality factor, and $\det A$ could be divergent and require renormalization.

Let us now compute $Z_E[J]$ for the case of the free scalar field:

$$S_E = \frac{1}{2} \int ((\partial_\mu \phi)^2 + m^2 \phi^2) d^D x = \frac{1}{2} \int \phi(x) (-\nabla^2 + m^2) \phi(x) d^D x = \frac{1}{2} \int \phi(x) \delta(x - y) (-\nabla_y^2 + m^2) \phi(y) d^D x d^D y$$

$$\text{hence } A(x, y) = \delta(x - y) (m^2 - \nabla_y^2)$$

$$\delta(x - z) = \int A(x, y) A^{-1}(y, z) d^D y = \int \delta(x - y) (-\nabla_y^2 + m^2) A^{-1}(y, z) d^D y = (-\nabla_x^2 + m^2) A^{-1}(x, z)$$

Note that $A(x, y)$ depends in fact only on $x - y$ (translation invariance), thus also $A^{-1}(x, z)$ depends only on $x - z$.

$$\Delta_E(x, z) \equiv A^{-1}(x, z), \quad \Delta_E(x - z) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-z)} \tilde{\Delta}_E(k), \quad \delta(x - z) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-z)}$$

$\rightarrow$ with our convention $kx = k^\mu x^\mu = \vec{k} \cdot \vec{x}$

$$\text{hence } (k^2 + m^2) \tilde{\Delta}_E(k) = 1 \text{ and}$$

$$\Delta_E(x, y) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} e^{-ik(x-y)}$$

Note that $\Delta_E(x, y) = \Delta_E(y, x)$, as follows from using $k \rightarrow -k$ in the integral

From Gaussian integration formulae it follows that

$$Z_E[J] = \exp\left\{\frac{1}{2} \int J(\alpha) \Delta_E(\alpha, \beta) J(\beta) d^D \alpha d^D \beta\right\}$$

Taking functional derivatives we have (using $\Delta_E(\alpha, \beta) = \Delta_E(\beta, \alpha)$)

$$\begin{aligned} \frac{\delta Z_E}{\delta J(x)} &= \exp\left\{\frac{1}{2} \int J(\alpha) \Delta_E(\alpha, \beta) J(\beta) d^D \alpha d^D \beta\right\} \left(\frac{1}{2} \int \Delta_E(x, \beta) J(\beta) d^D \beta + \frac{1}{2} \int J(\alpha) \Delta_E(\alpha, x) d^D \alpha\right) = \\ &= \exp\left\{\frac{1}{2} \int J(\alpha) \Delta_E(\alpha, \beta) J(\beta) d^D \alpha d^D \beta\right\} \int \Delta_E(x, \alpha) J(\alpha) d^D \alpha \end{aligned}$$

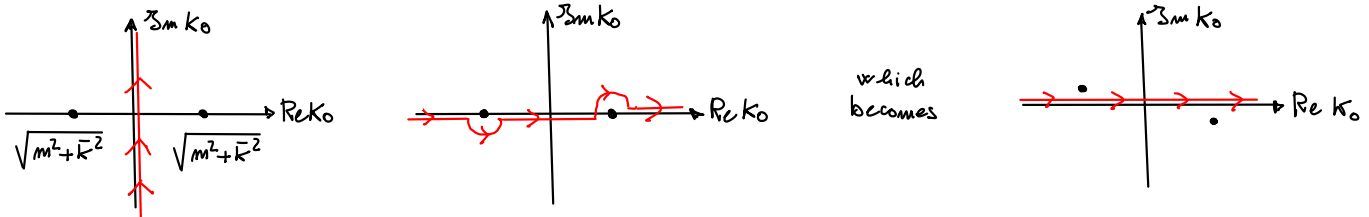
hence

$$\langle \phi(x) \phi(y) \rangle = \frac{\delta^2 Z_E}{\delta S(x) \delta S(y)} \Big|_{S=0} = \Delta_E(x, y) \quad \text{and } \Delta_E(x, y) \text{ is the Euclidean two point function}$$

Let us now rotate back to real (Minkowski) time, using $x^0 = -ix_4$, $k^0 = ik_4$, hence

$$kx = k_4 x_4 - \vec{k} \cdot \vec{x} = k_0 x_0 - \vec{k} \cdot \vec{x} = k_\mu x^\mu, \quad k^2 = k_4^2 + \vec{k}^2 = -k_0^2 + \vec{k}^2 = -k_\mu k^\mu \quad \text{and } d^D k = dk_4 d^3 k = -i dk_0 d^3 k$$

When rotating to real time we have to pay attention to the poles in $k_0^2 = m^2 + \vec{k}^2$ which appears in $\tilde{\Delta}_E(k)$, hence the integration path has to be deformed as follows



$$\text{by infinitesimally shifting the poles, i.e. by using } \Delta_E(x) = i \int \frac{e^{-ik_\mu x^\mu}}{k_\mu k^\mu - m^2 + i\epsilon} \frac{d^D k}{(2\pi)^D} \equiv i \Delta(x) \text{ where } \epsilon = 0^+.$$

Using the analytic continuation $i\Delta(x)$ of $\Delta_E(x)$ and $dx_4 = i dx^0$, $dy_4 = i dy^0$ we can continue from Euclidean to Minkowski the generating functional of free fields:

$$\int S(x) \Delta_E(x, y) S(y) d^D x d^D y \rightarrow -i \int S(x) \Delta(x, y) S(y) d^D x d^D y \quad \text{hence}$$

$$Z[S] = \exp \left\{ -\frac{i}{2} \int S(x) \Delta(x, y) S(y) d^D x d^D y \right\}$$

and, as a consequence, we obtain the propagator

$$\langle 0 | T[\phi(x) \phi(y)] | 0 \rangle = \frac{1}{i^2} \frac{\delta^2 Z}{\delta S(x) \delta S(y)} \Big|_{S=0} = i \Delta(x, y)$$

Wick theorem

In the finite dimensional (Euclidean) case the generating functional is $Z_E[b] = \exp \left\{ \frac{1}{2} \sum_{\alpha\beta} b_\alpha (A^{-1})_{\alpha\beta} b_\beta \right\}$ hence the n -point functions can be computed as

$$\langle x_{k_1} \dots x_{k_n} \rangle = \frac{\int \prod_{j=1}^N dx_j \prod_{i=1}^n x_{k_i} e^{-\frac{1}{2} \sum_{\alpha\beta} x_\alpha A_{\alpha\beta} x_\beta}}{\int \prod_{j=1}^N dx_j e^{-\frac{1}{2} \sum_{\alpha\beta} x_\alpha A_{\alpha\beta} x_\beta}} = \left(\prod_{i=1}^n \frac{\partial}{\partial b_{k_i}} \right) Z_E[b] \Big|_{b=0} = \left(\prod_{i=1}^n \frac{\partial}{\partial b_{k_i}} \right) e^{\frac{1}{2} \sum_{\alpha\beta} b_\alpha (A^{-1})_{\alpha\beta} b_\beta} \Big|_{b=0}$$

and it is simple to understand that $\langle x_{k_1} \dots x_{k_n} \rangle = 0$ if n is odd, and if n is even we have

$$\langle x_{k_1} \dots x_{k_n} \rangle = \sum_{\substack{\text{on all} \\ \text{couplings in} \\ k_1, \dots, k_n}} (A^{-1})_{k_{p_1} k_{p_2}} \dots (A^{-1})_{k_{p_{n-1}} k_{p_n}} \quad \text{which is the finite dimensional version of Wick theorem}$$

$$\text{In Euclidean field theory } \langle \phi(x_{k_1}) \dots \phi(x_{k_n}) \rangle = \sum_{\substack{\text{on all} \\ \text{couplings in} \\ k_1, \dots, k_n}} \Delta_E(x_{k_{p_1}}, x_{k_{p_2}}) \dots \Delta_E(x_{k_{p_{n-1}}}, x_{k_{p_n}})$$

and in Hinkowski $\langle 0|T[\phi(x_{k_1})\dots\phi(x_{k_n})]|0\rangle = \sum_{\substack{\text{on all} \\ \text{couplings in} \\ k_1, \dots, k_n}} i\Delta(x_{k_{p_1}}, x_{k_{p_2}}) \dots i\Delta(x_{k_{p_{m-1}}}, x_{k_{p_m}})$

PERTURBATION THEORY

Let us introduce the short-hand $\phi K \phi = \int \phi(x) K(x, y) \phi(y) d^D x d^D y$, $\int \phi = \int \phi(x) d^D x$, $V(\phi) = \frac{g}{4!} \int \phi^4(x) d^D x$

hence $Z_E[\int \phi] = \frac{\int \phi K \phi e^{-\frac{1}{2} \phi K \phi - V(\phi) + \int \phi}}{\int \phi K \phi e^{-\frac{1}{2} \phi K \phi - V(\phi)}}$ Using $\frac{\delta}{\delta \int \phi} e^{\int \phi} = \phi(x) e^{\int \phi}$ and $e^{-V(\phi)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} V(\phi)^m$

we have $\int \phi K \phi e^{-\frac{1}{2} \phi K \phi - V(\phi) + \int \phi} = \int \phi K \phi \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} V(\phi)^m e^{-\frac{1}{2} \phi K \phi + \int \phi} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} V\left[\frac{\delta}{\delta \int \phi}\right]^m \int \phi K \phi e^{-\frac{1}{2} \phi K \phi + \int \phi}$
 $= e^{-V[\delta/\delta \int \phi]} \int \phi K \phi e^{-\frac{1}{2} \phi K \phi + \int \phi}$ and using analogous manipulations on the denominator we have

$$Z_E[\int \phi] = \frac{e^{-V[\delta/\delta \int \phi]} \int \phi K \phi e^{-\frac{1}{2} \phi K \phi + \int \phi}}{\left[e^{-V[\delta/\delta \int \phi]} \int \phi K \phi e^{-\frac{1}{2} \phi K \phi + \int \phi} \right]_{\int \phi=0}} = \frac{e^{-V[\delta/\delta \int \phi]} e^{\frac{1}{2} \Delta_E \int \phi}}{\left[e^{-V[\delta/\delta \int \phi]} e^{\frac{1}{2} \Delta_E \int \phi} \right]_{\int \phi=0}}$$

where in the second step we used the Gaussian integral of the previous section and the facts that prefactors are the same for numerator and denominator.

Any correlation function can thus be computed using

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{\delta Z_E[\int \phi]}{\delta \int \phi(x_1) \dots \delta \int \phi(x_n)} \Big|_{\int \phi=0} = \frac{\delta}{\delta \int \phi(x_1)} \dots \frac{\delta}{\delta \int \phi(x_n)} \frac{e^{-V[\delta/\delta \int \phi]} e^{\frac{1}{2} \Delta_E \int \phi}}{\left[e^{-V[\delta/\delta \int \phi]} e^{\frac{1}{2} \Delta_E \int \phi} \right]_{\int \phi=0}} \Big|_{\int \phi=0}$$

and by expanding e^{-V} in numerator and denominator we get the perturbative expansion, whose computation boils down to the computation of functional derivatives of Gaussian generating functionals, something that is equivalent to the use of Wick theorem. Any perturbative contribution is a sum of products of propagators, integrated on all points corresponding to the interaction term $V(\phi) = \frac{g}{4!} \int \phi^4(x) d^D x$, which can be represented by Feynman diagrams

Let us consider the computation of $\langle \phi(x_1) \phi(x_2) \rangle$

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{\frac{\delta}{\delta \int \phi(x_1)} \frac{\delta}{\delta \int \phi(x_2)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\partial}{\partial 4!} \right)^m \int \prod_{i=1}^m d^D y_i \prod_{i=1}^m \left(\frac{\delta}{\delta \int \phi(y_i)} \right)^4 e^{\frac{1}{2} \Delta_E \int \phi}}{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\partial}{\partial 4!} \right)^m \int \prod_{i=1}^m d^D y_i \prod_{i=1}^m \left(\frac{\delta}{\delta \int \phi(y_i)} \right)^4 e^{\frac{1}{2} \Delta_E \int \phi}} \Big|_{\int \phi=0}$$

and let us consider, to start, just contributions coming from the numerator.

$$n=0) \quad \frac{\delta}{\delta S(x_1)} \frac{\delta}{\delta S(x_2)} e^{\frac{1}{2} S \Delta_E S} \Big|_{S=0} = \Delta_E(x_1 - x_2) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik(x_1 - x_2)}}{k^2 + m^2} = \text{---} \text{---}$$

$$n=1) \quad \text{we have to compute} \quad - \frac{g}{4!} \frac{\delta}{\delta S(x_1)} \frac{\delta}{\delta S(x_2)} \int d^D y \left(\frac{\delta}{\delta S(y)} \right)^4 e^{\frac{1}{2} S \Delta_E S} \Big|_{S=0}$$

and we have to connect in

this generates the vertex $\times y$

all possible ways



$$\text{---} \text{---} \text{---} \text{---} = - \frac{4 \cdot 3}{4!} g \int d^D y \Delta_E(x_1 - y) \Delta_E(0) \Delta_E(x_2 - y)$$

where $4 \cdot 3 = \#$ of different ways to build this graph

and $\frac{4 \cdot 3}{4!} = \frac{1}{2}$ is known as the symmetry factor of the graph

$$\text{---} \text{---} \text{---} \text{---} = - \frac{g}{8} \Delta_E(x_1 - x_2) \int d^D y \Delta_E^2(0)$$

$$n=2) \quad \text{we have to compute} \quad \frac{1}{2!} \left(\frac{g}{4!} \right)^2 \frac{\delta}{\delta S(x_1)} \frac{\delta}{\delta S(x_2)} \int d^D y_1 d^D y_2 \left(\frac{\delta}{\delta S(y_1)} \right)^4 \left(\frac{\delta}{\delta S(y_2)} \right)^4 e^{\frac{1}{2} S \Delta_E S} \Big|_{S=0}$$

and we thus have to

join in all possible ways (note that the factor $\frac{1}{2!}$ is always cancelled by the $2!$ permutations of vertices)

$$\text{---} \text{---} \text{---} \text{---} = \frac{g^2}{6} \int \Delta_E^3(y_1 - y_2) \Delta_E(x_1 - y_1) \Delta_E(x_2 - y_2) d^D y_1 d^D y_2$$

$$\text{---} \text{---} \text{---} \text{---} = \frac{g^2}{4!} \int \Delta_E(x_1 - y_1) \Delta_E(x_2 - y_1) \Delta_E^2(y_2 - y_1) \Delta_E(0) d^D y_1 d^D y_2$$

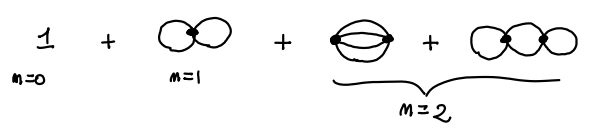
$$\text{---} \text{---} \text{---} \text{---} = \frac{g^2}{4} \int \Delta_E(x_1 - y_1) \Delta_E(y_1 - y_2) \Delta_E(y_2 - x_2) \Delta_E^2(0) d^D y_1 d^D y_2$$

$$\text{---} \text{---} \text{---} \text{---} = \frac{g^2}{48} \Delta_E(x_1 - x_2) \int \Delta_E^4(y_1 - y_2) d^D y_1 d^D y_2$$

$$\text{---} \text{---} \text{---} \text{---} = \frac{g^2}{16} \Delta_E(x_1 - x_2) \int \Delta_E^2(y_1 - y_2) \Delta_E^2(0) d^D y_1 d^D y_2$$

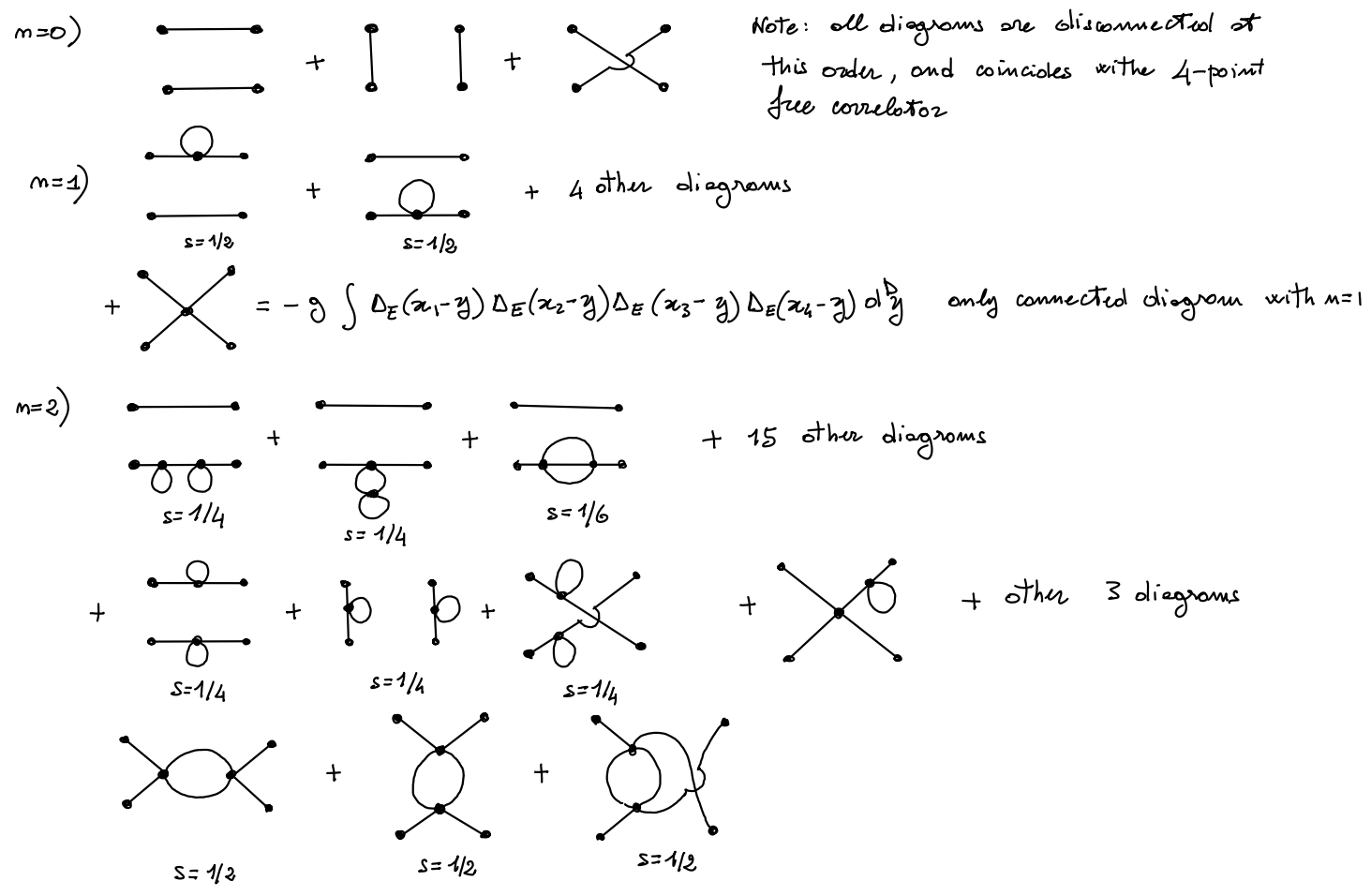
Note that, quite often, Feynman diagrams are associated to a single contraction scheme, and have thus to be multiplied a posteriori by symmetry factors.

By expanding the denominator of $\langle \phi(x_1) \phi(x_2) \rangle$ we get instead the "vacuum" (or "bubble") diagrams



and it is simple to check that all disconnected diagrams with bubbles of the numerator cancel in the ratio. This is a general fact, that will be proven below.

Let us now consider the computation of $\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle$, considering only the terms of the numerator which do not contain vacuum diagrams.



proof that vacuum bubbles cancel in the ratio which defines the perturbative expansion of n -point functions $\langle \phi(x_1) \dots \phi(x_n) \rangle$ at any order of the expansion. If we divide both numerator and denominator by $\int D\phi e^{-\frac{1}{2} \phi K \phi}$ and denote by $\langle \dots \rangle_0$ averages in the free field we have

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int \prod_{k=1}^j dy_k \langle \phi(x_1) \dots \phi(x_n) V(y_1) \dots V(y_j) \rangle_0}{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int \prod_{k=1}^j dy_k \langle V(y_1) \dots V(y_j) \rangle_0}$$

We can now use the Wick theorem to compute free field average values, and

$\langle \phi(x_1) \dots \phi(x_m) V(y_1) \dots V(y_j) \rangle_0$ is expanded as a sum of products of the form $\langle \phi(x_1) \dots \phi(x_m) V(y_1) \dots V(y_m) \rangle_{0, \text{NB}} \langle V(y_{m+1}) \dots V(y_j) \rangle_0$, where no vacuum bubbles are present in the first term. We have $\binom{j}{m} = \frac{j!}{m!(j-m)!}$ terms in the sum for each j and m , hence

$$\begin{aligned} \langle \phi(x_1) \dots \phi(x_m) \rangle &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int \prod_{k=1}^j d\mathbf{y}_k^D \langle V(y_1) \dots V(y_j) \rangle_0 = \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{m=0}^j \binom{j}{m} \int \prod_{k=1}^m d\mathbf{y}_k^D \langle \phi(x_1) \dots \phi(x_m) V(y_1) \dots V(y_m) \rangle_{0, \text{NB}} \int \prod_{k=m+1}^j d\mathbf{y}_k^D \langle V(y_{m+1}) \dots V(y_j) \rangle_0 \\ &= \sum_{j=0}^{\infty} \sum_{m=0}^j \frac{(-1)^m}{m!} \int \prod_{k=1}^m d\mathbf{y}_k^D \langle \phi(x_1) \dots \phi(x_m) V(y_1) \dots V(y_m) \rangle_{0, \text{NB}} \frac{(-1)^{j-m}}{(j-m)!} \int \prod_{k=m+1}^j d\mathbf{y}_k^D \langle V(y_{m+1}) \dots V(y_j) \rangle_0 \\ \text{and } \sum_{j=0}^{\infty} \sum_{m=0}^j &= \begin{array}{|c|} \hline j \\ \hline m \\ \hline \end{array} = \begin{array}{|c|} \hline j \\ \hline m \\ \hline \end{array} = \sum_{m=0}^{\infty} \sum_{j=m}^{\infty} \text{ and the sum } \sum_{j=m}^{\infty} \text{ just gives the denominator,} \end{aligned}$$

thus $\langle \phi(x_1) \dots \phi(x_m) \rangle = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int \prod_{k=1}^m d\mathbf{y}_k^D \langle \phi(x_1) \dots \phi(x_m) V(y_1) \dots V(y_m) \rangle_{0, \text{NB}}$

GENERATING FUNCTIONALS

We defined the generating functional of correlation functions by (in Euclidean time)

$$\mathcal{Z}[\mathcal{J}] = \frac{\int \mathcal{D}\phi e^{-S[\phi] + \int \mathcal{J}\phi}}{\int \mathcal{D}\phi e^{-S[\phi]}}$$

but we have seen that several terms of the perturbative expansion can be written as products of lower order terms, so we want to isolate just the "minimal" part that we really have to compute.

The part of the n -point correlation function $G_n(x_1, \dots, x_n)$ which can not be written as (sum of) products of n' -point correlation functions with $n' < n$ is the connected n -point correlation function $W_n(x_1, \dots, x_n)$.

The 1-point function, whenever it does not vanish, it is connected, since the only disconnected terms are those involving vacuum bubbles, which cancel in the ratio of $\mathcal{Z}[\mathcal{J}]$. Starting from

$W_1(x_1) = G_1(x_1)$ we can recursively define the n -point connected functions by

$$G_n(x_1, \dots, x_n) = \sum_i \prod_i W_{\text{size}(I_i)}(I_i)$$

where the sum extends on the partitions I_i of $1, \dots, n$. The first few cases are

$$G_2(x_1, x_2) = W_2(x_1, x_2) + W_1(x_1) W_1(x_2)$$

$$\begin{aligned} G_3(x_1, x_2, x_3) &= W_3(x_1, x_2, x_3) + W_2(x_1, x_2) W_1(x_3) + W_2(x_1, x_3) W_1(x_2) + W_2(x_2, x_3) W_1(x_1) + \\ &+ W_1(x_1) W_1(x_2) W_1(x_3) \end{aligned}$$

and, in general, $W_m(x_1, \dots, x_m) = G_m(x_1, \dots, x_m) - \text{disconnected terms}$. For several theories

1-point function vanishes (e.g. because of parity or gauge invariance) and expressions get simpler:

$$G_1(x_1) = W_1(x_1) = 0, \quad G_2(x_1, x_2) = W_2(x_1, x_2), \quad G_3(x_1, x_2, x_3) = W_3(x_1, x_2, x_3)$$

$$G_4(x_1, x_2, x_3, x_4) = W_4(x_1, x_2, x_3, x_4) + W_2(x_1, x_2)W_2(x_3, x_4) + W_2(x_1, x_3)W_2(x_2, x_4) + W_2(x_1, x_4)W_2(x_2, x_3)$$

For the ϕ^4 theory, diagrams that contribute to $W_4(x_1, x_2, x_3, x_4)$ up to second order are



If we define $W[\mathcal{J}] = \log Z[\mathcal{J}]$ it can be shown that $W[\mathcal{J}]$ is the generator of connected correlation functions (whose perturbative expansion just involve connected diagrams). Let us check explicitly the first few cases:

$$\frac{\delta W}{\delta \mathcal{J}(x_1)} = \frac{1}{Z[\mathcal{J}]} \frac{\delta Z[\mathcal{J}]}{\delta \mathcal{J}(x_1)} \quad \text{and using } Z[0] = 1 \quad W_1(x_1) = \left. \frac{\delta W}{\delta \mathcal{J}(x_1)} \right|_{\mathcal{J}=0} = \left. \frac{\delta Z}{\delta \mathcal{J}(x_1)} \right|_{\mathcal{J}=0} = G_1(x_1)$$

$$\frac{\delta^2 W}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2)} = \frac{1}{Z[\mathcal{J}]} \frac{\delta^2 Z}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2)} - \frac{1}{Z^2[\mathcal{J}]} \frac{\delta Z}{\delta \mathcal{J}(x_1)} \frac{\delta Z}{\delta \mathcal{J}(x_2)} \quad \text{hence } G_2(x_1, x_2) = W_2(x_1, x_2) + W_1(x_1)W_1(x_2)$$

$$\begin{aligned} \frac{\delta^3 W}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2) \delta \mathcal{J}(x_3)} &= \frac{1}{Z[\mathcal{J}]} \frac{\delta^3 Z}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2) \delta \mathcal{J}(x_3)} - \frac{1}{Z^2[\mathcal{J}]} \frac{\delta Z}{\delta \mathcal{J}(x_2)} \frac{\delta^2 Z}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_3)} - \frac{1}{Z^2[\mathcal{J}]} \frac{\delta^2 Z}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2)} \frac{\delta Z}{\delta \mathcal{J}(x_3)} \\ &\quad - \frac{1}{Z^2[\mathcal{J}]} \frac{\delta^2 Z}{\delta \mathcal{J}(x_2) \delta \mathcal{J}(x_3)} \frac{\delta Z}{\delta \mathcal{J}(x_1)} \end{aligned}$$

$$\text{and using } \left. \frac{1}{Z^2[\mathcal{J}]} \frac{\delta Z}{\delta \mathcal{J}(x_3)} \frac{\delta^2 Z}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2)} \right|_{\mathcal{J}=0} = G_1(x_3) G_2(x_1, x_2) = W_1(x_3) (W_2(x_1, x_2) + W_1(x_1)W_1(x_2))$$

and analogous relations one gets

$$G_3(x_1, x_2, x_3) = W_3(x_1, x_2, x_3) + W_1(x_1)W_2(x_2, x_3) + W_1(x_2)W_2(x_1, x_3) + W_1(x_3)W_2(x_1, x_2) + W_1(x_1)W_1(x_2)W_1(x_3)$$

hence we have verified that, at least up to 3-point functions, $W[\mathcal{J}] = \log Z[\mathcal{J}]$ is the generator of connected correlation functions.

We have introduced connected correlation functions by $G_m(x_1, \dots, x_m) = \sum_{\text{part. } i} \prod_i W(I_i)$ and it is simple to understand that if $I_i^{(j)}$ (for fixed i and $j=1, \dots, m_i$) are m_i partitions all with the same number k_i of elements, then

$$P[\mathcal{J}] = \frac{1}{m_i!} \left(\int \frac{1}{k_i!} W(I_i) \prod_{m \in I_i} \mathcal{J}(x_m) dx_m \right)^{m_i} \quad \text{is the generating functional for } \sum_j \prod_j W(I_i^{(j)})$$

where the sum extends on all partitions $I_i^{(j)}$ of the elements $1, \dots, m_i k_i$.

The important point to be noted is that $W_k(x_1, \dots, x_k)$ is a symmetric function, hence two partitions are equivalent if they differ only for the order in which elements enter $W(I_i^{(j)})$,

hence there is $1/k_i!$ for each $W(I_i)$. Moreover, also switching $W(I_i^{(1)})$ with $W(I_i^{(2)})$ does not change the result, which motivates the $1/m_i!$ factor.

An example is the following: it is simple to verify explicitly that

$$\frac{1}{2!} \left(\frac{1}{2!} \int W_2(\xi_1, \xi_2) \delta(\xi_1) \delta(\xi_2) d\xi_1 d\xi_2 \right)^2 = \frac{1}{(2!)^2} \int W_2(\xi_1, \xi_2) W_2(\eta_1, \eta_2) \delta(\xi_1) \delta(\xi_2) \delta(\eta_1) \delta(\eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

generates $W_2(x_1, x_2) W_2(x_3, x_4) + W_2(x_1, x_3) W_2(x_2, x_4) + W_2(x_1, x_4) W_2(x_2, x_3)$.

By extending the analysis to the case in which partitions of different sizes exist we can show that

$\prod_i \frac{1}{m_i!} \left(\frac{1}{k_i!} W(I_i) \prod_{m \in I_i} \delta(x_m) dx_m \right)^{m_i}$ generates $\sum_{ij} \prod W(I_i^{(j)})$ where the sum extends on the partitions of $n = \sum k_i m_i$ objects in which m_i partitions have k_i objects.

Hence $Z[\mathcal{J}] = \sum_n \sum_{\text{partitions of } x_1, \dots, x_n} \prod_i \frac{1}{m_i!} \left(\frac{1}{k_i!} \int W(I_i) \prod_{m \in I_i} \delta(x_m) dx_m \right)^{m_i} =$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \left(\int W(x_1) \delta(x_1) dx_1 + \frac{1}{2!} \int W(x_1, x_2) \delta(x_1) \delta(x_2) dx_1 dx_2 + \dots + \frac{1}{k!} \int W(x_1, \dots, x_k) \delta(x_1) \dots \delta(x_k) dx_1 \dots dx_k + \dots \right)^N = \exp W[\mathcal{J}]$$

where in the second step we used $\frac{1}{N!} (y_1 + \dots + y_k + \dots)^N = \frac{1}{N!} \sum \frac{N!}{m_1! \dots m_k! \dots} y_1^{m_1} \dots y_k^{m_k} \dots$

A similar proof can be carried out order by order in perturbation theory: the N -th order has a factor $\frac{1}{N!}$ and we have to sum on all contractions (equivalent to a sum on all partitions of N elements). If the contraction consists of m_1 connected diagrams with k_1 vertices, m_2 connected diagrams with k_2 vertices and so on (hence $N = \sum m_i k_i$), the number of independent ways of getting it is $\frac{N!}{\prod_i [(k_i!)^{m_i} m_i!]}$ and we can use manipulations similar to those used above.

Analogous manipulations are used in the polymer expansion in constructive quantum field theory, see GLIMM, SAFFE "Quantum physics, a functional integral point of view" § 30, and the previous result is in fact a particular case of the exponential formula in combinatorics, see, e.g. STANLEY "Enumerative combinatorics" vol. 2 § 5.

For a non-combinatoric proof that $W[\mathcal{J}]$ is the generator of connected correlation functions see ZINN-SUSTIN "Quantum field theory and critical phenomena" § 7.4.1

To continue this result to real (Minkowsky) time we have to use $\delta(x_k) dx_k \rightarrow i \delta(x_k) dx_k$ and it is immediate to understand that the generator of connected correlators remains $W[\mathcal{J}] = \log Z[\mathcal{J}]$.

Let us introduce the notation $\varphi(x) = \frac{\delta W[\mathcal{J}]}{\delta \mathcal{J}(x)}$ (not at $\mathcal{J}=0$) where $\mathcal{J}$ is to be thought as an external classical source.

The effective action $\Gamma[\varphi]$ is defined as the Legendre transform of $W[\mathcal{J}]$:

$$\Gamma[\varphi] = \int \mathcal{J}(x) \varphi(x) d^D x - W[\mathcal{J}] \quad \text{where } \mathcal{J}(x) \text{ has to be written as a function of } \varphi(x).$$

By considering a derivative with respect to φ we get

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = \mathcal{J}(x) + \int \frac{\delta \mathcal{J}(y)}{\delta \varphi(x)} \frac{\delta}{\delta \mathcal{J}(y)} \left[\int \mathcal{J} \varphi - W[\mathcal{J}] \right]_{\varphi \text{ const}} d^D y = \mathcal{J}(x); \quad \text{hence} \quad \boxed{\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = \mathcal{J}(x)}$$

$\varphi(y) - \varphi(y) = 0$

and the Legendre transform is involutive.

Analogously, if $W[\mathcal{J}]$ depends on the parameter λ we have

$$\left. \frac{\partial \Gamma}{\partial \lambda} \right|_{\varphi \text{ const}} = - \left. \frac{\partial W}{\partial \lambda} \right|_{\mathcal{J} \text{ const}} + \int \frac{\partial \mathcal{J}(y)}{\partial \lambda} \frac{\delta}{\delta \mathcal{J}(y)} \left[\int \mathcal{J} \varphi - W[\mathcal{J}] \right]_{\varphi \text{ const}} d^D y = - \left. \frac{\partial W}{\partial \lambda} \right|_{\mathcal{J} \text{ const}}$$

$$\text{hence} \quad \left. \frac{\partial \Gamma}{\partial \lambda} \right|_{\varphi \text{ const}} + \left. \frac{\partial W}{\partial \lambda} \right|_{\mathcal{J} \text{ const}} = 0.$$

Clearly $\varphi(x)|_{\mathcal{J}=0} = \left. \frac{\delta W}{\delta \mathcal{J}} \right|_{\mathcal{J}=0} = W_1(x) = \langle \phi(x) \rangle$, and from

$$W[\mathcal{J}] = \int W_1(x_1) \mathcal{J}(x_1) d^D x_1 + \frac{1}{2!} \int W_2(x_1, x_2) \mathcal{J}(x_1) \mathcal{J}(x_2) d^D x_1 d^D x_2 + \dots + \frac{1}{n!} \int W_n(x_1, \dots, x_n) \prod_{i=1}^n \mathcal{J}(x_i) d^D x_i + \dots$$

we get (using the symmetry of $W_n(x_1, \dots, x_n)$)

$$\varphi(x) = \langle \phi(x) \rangle + \int W_2(x, y_1) \mathcal{J}(y_1) d^D y_1 + \dots + \frac{1}{(n-1)!} \int W_n(x, y_1, \dots, y_{n-1}) \prod_{i=1}^{n-1} \mathcal{J}(y_i) d^D y_i + \dots \quad (*)$$

and if we introduce $\eta(x) = \varphi(x) - \langle \phi(x) \rangle$

we can iteratively solve the previous equation, to get $\mathcal{J}(x)$ as a function of $\eta(x)$. If we introduce

$$S(x, y) \quad \text{by} \quad \int S(x, y) W_2(y, z) d^D y = \delta(x - z) \quad \text{it is immediate to see that at the first order we have}$$

$$\mathcal{J}(x) = \int S(x, y) \eta(y) d^D y$$

$$\text{If we write for the next order} \quad \mathcal{J}(x) = \int S(x, y_1) \eta(y_1) d^D y_1 + \frac{1}{2!} \int F(x, y_1, y_2) \eta(y_1) \eta(y_2) d^D y_1 d^D y_2$$

and substitute in (*) we get

$$0 = \frac{1}{2!} \int W_2(x, z_1) F(z_1, y_1, y_2) \eta(y_1) \eta(y_2) d^D z_1 d^D y_1 d^D y_2 + \frac{1}{2!} \int W_3(x, x_1, x_2) S(x_1, y_1) S(x_2, y_2) \eta(y_1) \eta(y_2) d^D x_1 d^D x_2 d^D y_1 d^D y_2$$

from which

$$F(x_1, x_2, x_3) = - \int W(y_1, y_2, y_3) S(y_1, x_1) S(y_2, x_2) S(y_3, x_3) d^D y_1 d^D y_2 d^D y_3$$

If we introduce the connected amputated n-point functions by

$$W_{n, \text{amp}}(x_1, \dots, x_n) = \int W_n(y_1, \dots, y_n) \prod_{i=1}^n (S(y_i, x_i) d^D y_i)$$

we thus have to second order

$$\mathcal{J}(x) = \int S(x, y_1) \eta(y_1) d^D y_1 - \frac{1}{2} \int W_{3, \text{amp}}(x, y_1, y_2) \eta(y_1) \eta(y_2) d^D y_1 d^D y_2$$

and so on. Since $\mathcal{J}(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)}$, the coefficient obtained by this iterative procedure

are the coefficients of the functional Taylor expansion of the effective action:

$$\Gamma[\varphi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \Gamma_n(x_1, \dots, x_n) \varphi(x_1) \dots \varphi(x_n) d^4x_1 \dots d^4x_n$$

$\Gamma_1(x_1)=0$; $\Gamma_2(x_1, x_2) = S(x_1, x_2)$; $\Gamma_3(x_1, x_2, x_3) = -W_{3, \text{amp}}(x_1, x_2, x_3)$

and so on. There is also a different way of computing these coefficients: we have defined

$\varphi(x) = \frac{\delta W}{\delta S(x)}$ and we have shown that $S(x) = \frac{\delta \Gamma}{\delta \varphi(x)}$, hence

$$\delta(x_1 - x_2) = \frac{\delta^2 \Gamma}{\delta \varphi(x_1) \delta S(x_2)} = \int d^4y \frac{\delta^2 \Gamma}{\delta \varphi(x_1) \delta \varphi(y)} \frac{\delta \varphi(y)}{\delta S(x_2)} = \int d^4y \frac{\delta^2 \Gamma}{\delta \varphi(x_1) \delta \varphi(y)} \frac{\delta^2 W}{\delta S(y) \delta S(x_2)} \quad *$$

Evaluating this expression at $S=0$ (hence $\varphi = \langle \phi \rangle$ or $\eta=0$) we get $\Gamma_3(x_1, x_2) = S(x_1, x_2)$. If instead

we take the further derivative $\delta/\delta \varphi(x_3)$ we find

$$0 = \int d^4y \frac{\delta^3 \Gamma}{\delta \varphi(x_1) \delta \varphi(y) \delta \varphi(x_3)} \frac{\delta^2 W}{\delta S(y) \delta S(x_2)} + \int d^4y_1 d^4y_2 \frac{\delta^2 \Gamma}{\delta \varphi(x_1) \delta \varphi(y_1)} \frac{\delta^3 W}{\delta S(y_1) \delta S(x_2) \delta S(y_2)} \frac{\delta^2 \Gamma}{\delta \varphi(y_2) \delta \varphi(x_3)}$$

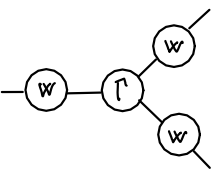
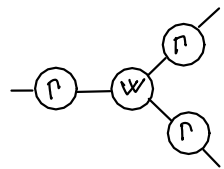
from which one easily sees that $\Gamma_3 = -W_{3, \text{amp}}$. To go on it is convenient to introduce

the notation  for $\frac{\delta^m \Gamma}{\delta \varphi(x_1) \dots \delta \varphi(x_m)}$ and  for $\frac{\delta^m W}{\delta S(x_1) \dots \delta S(x_m)}$. Using

these notations the previous equation can be rewritten as

$$\begin{array}{c} x_1 \\ \nearrow \\ \Gamma \\ \nwarrow \\ x_3 \end{array} \begin{array}{c} y \\ \longrightarrow \end{array} \begin{array}{c} W \\ \longrightarrow \end{array} x_2 = - \begin{array}{c} x_1 \longrightarrow \Gamma \longrightarrow y_1 \longrightarrow W \begin{array}{c} \nearrow x_2 \\ \searrow \Gamma \searrow x_3 \end{array} \end{array}$$

which can be rewritten as (see *)

*²  = -  or  = -  i.e. $\Gamma_3 = -W_{3, \text{amp}}$

and higher derivatives can be obtained graphically by using

$$\frac{\delta}{\delta S(x_{m+1})} \begin{array}{c} x_1 \\ \vdots \\ W \\ \vdots \\ x_m \end{array} = \begin{array}{c} x_1 \\ \vdots \\ W \\ \vdots \\ x_{m+1} \end{array} \quad \frac{\delta}{\delta S(x_{m+1})} \begin{array}{c} x_1 \\ \vdots \\ \Gamma \\ \vdots \\ x_m \end{array} = \begin{array}{c} x_1 \\ \vdots \\ \Gamma \\ \vdots \\ x_m \end{array} \begin{array}{c} y \\ \longrightarrow \end{array} \begin{array}{c} W \\ \longrightarrow \end{array} x_{m+1}$$

and the equations obtained in previous steps:

$$\begin{array}{c} \diagup \\ W \\ \diagdown \end{array} = - \begin{array}{c} \diagup W \\ \diagdown W \end{array} \begin{array}{c} \Gamma \\ \longrightarrow \end{array} \begin{array}{c} \diagup W \\ \diagdown W \end{array} - \begin{array}{c} \diagup W \\ \diagdown \end{array} \begin{array}{c} \Gamma \\ \longrightarrow \end{array} \begin{array}{c} \diagup W \\ \diagdown W \end{array} - \begin{array}{c} \diagup W \\ \diagdown \end{array} \begin{array}{c} \Gamma \\ \longrightarrow \end{array} \begin{array}{c} \diagup W \\ \diagdown W \end{array} - \begin{array}{c} \diagup W \\ \diagdown \end{array} \begin{array}{c} \Gamma \\ \longrightarrow \end{array} \begin{array}{c} \diagup W \\ \diagdown W \end{array}$$

and using *² we have

$$- \begin{array}{c} \diagup W \\ \diagdown \end{array} \begin{array}{c} \Gamma \\ \longrightarrow \end{array} \begin{array}{c} \diagup W \\ \diagdown W \end{array} = \begin{array}{c} \diagup W \\ \diagdown \end{array} \begin{array}{c} \Gamma \\ \longrightarrow \end{array} \begin{array}{c} \diagup W \\ \diagdown \end{array} \begin{array}{c} \Gamma \\ \longrightarrow \end{array} \begin{array}{c} \diagup W \\ \diagdown W \end{array}$$

It should be clear, and it can be proven by induction, that the n -point correlation function can be written as a sum of effective diagrams in which propagators are W_2 , vertices are $-T_n$ and only tree diagram has to be used. Note that these "diagrams" are not generated order by order in perturbation theory, but are all-orders diagrams.

If we consider the perturbative expansion of W_n ($\times + \text{loop} + \text{other}$), the previous result shows that if we are interested to T_n we do not have to take into account those diagrams which are generated by the contraction with W_2 , i.e. those which becomes disconnected if we remove a propagator. To compute T_n we just need diagrams which remains connected after removing a propagator, the so called one particle irreducible (1PI) diagrams.

The same is true for all T_n with $n \geq 2$, and also for T_2 , but in this last case we need a different proof. For the free theory $W_2(x, y) = \Delta_E(x, y) = k^{-1}(x, y)$, where k is the quadratic kernel of the free theory, hence $T_2(x, y) = W_2^{-1}(x, y) = k(x, y)$. For an interacting theory let us define $\Sigma(x, y)$, the so called self energy, by the relation $T_2(x, y) = k(x, y) + \Sigma(x, y)$; the self-energy takes into account all the perturbative corrections. Since $T_2 = W_2^{-1}$ we have $W_2 = (k + \Sigma)^{-1} = [(1 + \Sigma k^{-1}) k]^{-1} = k^{-1} (1 + \Sigma k^{-1})^{-1} = k^{-1} (1 - \Sigma k^{-1} + \Sigma k^{-1} \Sigma k^{-1} - \Sigma k^{-1} \Sigma k^{-1} \Sigma k^{-1} + \dots)$ hence $-\Sigma$ is the sum of all 2 point connected 1PI diagrams with the external propagators removed

$$-\Sigma(x, y) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

In Minkowsky time the effective action can be defined by using $\varphi(x) = \frac{1}{i} \frac{\delta W}{\delta J(x)}$ and

$$i\Gamma[\varphi] = W[J] - i \int J(x) \varphi(x) d^D x, \text{ so that } i \frac{\delta \Gamma}{\delta \varphi(x)} = -i J(x) \text{ and}$$

$$\delta(x-y) = \frac{\delta J(x)}{\delta J(y)} = - \frac{\delta^2 \Gamma}{\delta \varphi(x) \delta \varphi(y)} = - \int \frac{\delta^2 \Gamma}{\delta \varphi(x) \delta \varphi(z)} \frac{\delta \varphi(z)}{\delta J(y)} d^D z = +i \int \frac{\delta^2 \Gamma}{\delta \varphi(x) \delta \varphi(z)} \frac{\delta^2 W}{\delta J(z) \delta J(y)} d^D z \text{ and so on.}$$

SEMICLASSICAL EXPANSION

If we do not use $\hbar=1$ the generating functional of n -point functions is given by

$$Z[J] = \frac{\int D\phi \exp \left\{ -\frac{1}{\hbar} (S[\phi] - \int J(x) \phi(x) d^D x) \right\}}{\int D\phi \exp \left\{ -S[\phi]/\hbar \right\}}$$

we will use the shorthand $\int \phi = \int \phi(x) d^D x$,

and in the following we will use for concreteness the (Euclidean) action

$$S[\phi] = \frac{1}{2} \phi k \phi + V(\phi) = \frac{1}{2} \int \phi(x) k(x, y) \phi(y) d^D x d^D y + \frac{g}{4!} \int \phi^4(x) d^D x \quad k(x, y) = \delta(x-y) (-\nabla^2 + m^2)$$

In the limit $\hbar \rightarrow 0$ the path-integral can be approximated by using the leading term of the asymptotic expansion generated by the Laplace method.

Laplace method for 1dim integrals: $I_\alpha = \int_{-\infty}^{+\infty} e^{\alpha f(x)} dx$ and let us assume that $f'(x)=0$ has only a solution, $\bar{x}$, with moreover $f''(\bar{x}) < 0$, the only stationary point of $f(x)$ is a maximum. Then, in the limit $\alpha \rightarrow \infty$, we can approximate

$$I_\alpha \simeq \int_{-\infty}^{+\infty} \exp \left\{ \alpha \left[f(\bar{x}) + \frac{1}{2} f''(\bar{x}) (x - \bar{x})^2 \right] \right\} dx = e^{\alpha f(\bar{x})} \sqrt{-\frac{2\pi}{\alpha f''(\bar{x})}}$$

and the dominant contribution is $I_\alpha \sim e^{\alpha f(\bar{x})}$. For more details see, e.g. ERDELYI "Asymptotic expansions" § 35,36 or BENDER, ORSZAG "Advanced mathematical methods for scientists and engineers" § 6.4

Using the leading term of the Laplace method, and neglecting factors independent of $\mathcal{J}$

we have $\mathcal{Z}[\mathcal{J}] \simeq \exp \left\{ -\frac{1}{\hbar} (\mathcal{S}[\phi_c] - \mathcal{J}\phi_c) \right\}$, where ϕ_c is the solution of

$$\frac{\delta \mathcal{S}[\phi]}{\delta \phi(x)} = \mathcal{J}(x) \quad , \quad \text{i.e.,} \quad K\phi_c + \frac{\delta V}{\delta \phi(x)}[\phi_c] = \mathcal{J}$$

which can be rewritten, using $K^{-1} = \Delta_E$, in the form $\phi_c = \Delta_E \mathcal{J} - \Delta_E \frac{\delta V}{\delta \phi}[\phi_c]$, which is the integral equation

$$\phi_c(x) = \int \Delta_E(x,y) \mathcal{J}(y) dy - \frac{\partial}{\partial!} \int \Delta_E(x,y) \phi_c^3(y) dy$$

This equation can be solved iteratively in perturbation theory. The integral equation can

be represented graphically as $\phi_c = \text{---} \mathcal{J} - \frac{\partial}{\partial!} \text{---} \phi_c^3$

and the perturbative solution can be written as

$$\phi_c = \text{---} \mathcal{J} - \frac{\partial}{\partial!} \text{---} \mathcal{J}^3 + \left(\frac{\partial}{\partial!} \right)^2 \left[\text{---} \mathcal{J}^5 + \text{---} \mathcal{J}^4 \mathcal{J} + \text{---} \mathcal{J}^3 \mathcal{J}^2 \right] + \dots$$

in which only tree diagrams enter.

At the lowest order in $\hbar$ we thus have $\bar{\mathcal{Z}}_{(0)}[\mathcal{J}] \propto \exp \left\{ -\frac{1}{\hbar} (\mathcal{S}[\phi_c] - \mathcal{J}\phi_c) \right\}$, where ϕ_c can be determined using the above perturbative expansion.

When $\hbar \neq 1$ the generating functional of connected correlation functions is given by

$\mathcal{W}[\mathcal{J}] = \hbar \log \mathcal{Z}[\mathcal{J}]$, so at the lowest order in $\hbar$ we have

$$\mathcal{W}_{(0)}[\mathcal{J}] = \hbar \log \bar{\mathcal{Z}}_{(0)}[\mathcal{J}] = -\mathcal{S}[\phi_c] + \mathcal{J}\phi_c$$

$$\text{and from} \quad \frac{\delta \mathcal{W}_{(0)}}{\delta \mathcal{J}(x)} = \phi_c(x) + \int dy \frac{\delta \phi_c(y)}{\delta \mathcal{J}(x)} \underbrace{\frac{\delta}{\delta \phi_c(y)} [-\mathcal{S}[\phi_c] + \mathcal{J}\phi_c]}_{=0} = \phi_c(x)$$

we have $\varphi(x) = \frac{\delta \mathcal{W}_{(0)}}{\delta \mathcal{J}(x)} = \phi_c(x)$ at this order.

The effective action at the lowest order in $\hbar$ is thus given by

$$\Gamma_{(0)}[\varphi] = -W_{(0)}[\mathcal{J}] + \mathcal{J}\varphi = S[\phi_c] \quad \text{The effective action at order } \hbar^0 \text{ coincides with the classical action}$$

and using the development

$$\Gamma[\varphi] = \sum_{m=1}^{\infty} \frac{1}{m!} \int d^D x_1 \dots d^D x_m \Gamma_m(x_1, \dots, x_m) \eta(x_1) \dots \eta(x_m), \quad \text{with } \eta(x) = \varphi(x) - \langle \phi(x) \rangle, \text{ we find}$$

$$\Gamma_{(0)2}(x_1, x_2) = \delta(x_1 - x_2) (-\nabla^2 + m^2), \quad \Gamma_{(0)4}(x_1, \dots, x_4) = g \delta(x_1 - x_2) \delta(x_1 - x_3) \delta(x_1 - x_4), \quad \Gamma_{(0)m} = 0 \text{ se } m > 4$$

Let us now consider the semiclassical expansion at $O(\hbar)$: by writing $\phi = \phi_c + \hbar^{1/2} \chi$ we have

$$S[\phi] - \mathcal{J}\phi = S[\phi_c] - \mathcal{J}\phi_c + \int \hbar^{1/2} \chi(x) \left[\frac{\delta S}{\delta \phi(x)} - \mathcal{J}(x) \right]_{\phi_c} d^D x + \frac{\hbar}{2} \int \frac{\delta^2 S}{\delta \phi(x_1) \delta \phi(x_2)} \Big|_{\phi_c} \chi(x_1) \chi(x_2) d^D x_1 d^D x_2 + \dots$$

= 0 by def of ϕ_c

hence to this order we get

$$Z[\mathcal{J}] \propto Z_{(0)}[\mathcal{J}] \int d^D x \exp \left\{ -\frac{1}{2} \int x \frac{\delta^2 S}{\delta \phi \delta \phi} \Big|_{\phi_c} \chi \right\} \propto Z_{(0)}[\mathcal{J}] \left(\det \frac{\delta^2 S}{\delta \phi \delta \phi} \Big|_{\phi_c[\mathcal{J}]} \right)^{-1/2}$$

We can now compute $W[\mathcal{J}] = \hbar \log Z[\mathcal{J}] = W_{(0)}[\mathcal{J}] + \hbar W_{(1)}[\mathcal{J}]$ using $\log \det = \text{tr} \log$, obtaining

$$W_{(1)}[\mathcal{J}] = -\frac{1}{2} \text{tr} \left[\log \frac{\delta^2 S}{\delta \phi \delta \phi} \Big|_{\phi_c[\mathcal{J}]} - \log \frac{\delta^2 S}{\delta \phi \delta \phi} \Big|_{\phi_c[0]} \right] \quad \text{and the second term ensures } Z[0] = 1$$

When there is no spontaneous breaking of symmetry we have $\lim_{\mathcal{J} \rightarrow 0} \phi_c[\mathcal{J}] = 0$, hence $\phi_c[0] = 0$. If this does not happen, we can rewrite everything using $\bar{\phi}(x) = \phi(x) - \lim_{\mathcal{J} \rightarrow 0} \phi_c[\mathcal{J}]$, in such a way that $\bar{\phi}_c[0] = 0$.

For the ϕ^4 theory we have

$$\frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} = k(x, y) + \frac{g}{2} \phi^2(x) \delta(x - y)$$

and by using $\text{tr} \log(AB) = \log \det(AB) = \log[\det A \det B] = \text{tr} \log A + \text{tr} \log B$ we have

$$\begin{aligned} W_{(1)}[\mathcal{J}] &= -\frac{1}{2} \text{tr} \left[\log \left(k(x_1, x_2) + \frac{g}{2} \phi_c^2(x_1) \delta(x_1 - x_2) \right) - \log k(x_1, x_2) \right] = \\ &= -\frac{1}{2} \text{tr} \log \left(\delta(x_1 - x_2) + \frac{g}{2} \phi_c^2(x_1) \Delta_E(x_1, x_2) \right) \quad \left(\text{remember that } \int k(x_1, x_2) \Delta_E(x_2, x_3) d^D x_2 = \delta(x_1 - x_3) \right) \end{aligned}$$

Using the Taylor expansion $\log(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m$ we have

$$\begin{aligned} W_{(1)}[\mathcal{J}] &= -\frac{1}{2} \left[\frac{g}{2} \int d^D x_1 \phi_c^2(x_1) \Delta_E(x_1, x_1) - \frac{g}{8} \int d^D x_1 d^D x_2 \phi_c^2(x_1) \Delta_E(x_1, x_2) \phi_c^2(x_2) \Delta_E(x_2, x_1) + \dots \right. \\ &\quad \left. + \frac{(-1)^{n+1}}{n} g^n \int d^D x_1 \dots d^D x_n \phi_c^2(x_1) \Delta_E(x_1, x_2) \dots \phi_c^2(x_n) \Delta_E(x_n, x_1) + \dots \right] \end{aligned}$$

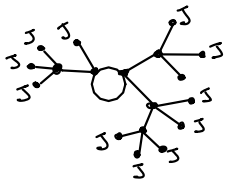
and to expose the dependence on $\mathcal{J}$ we have to substitute

$$\phi_c = \Delta_E \mathcal{J} - \Delta_E \frac{\delta V}{\delta \phi}(\phi_c) = \Delta_E \mathcal{J} - \Delta_E \frac{\delta V}{\delta \phi}(\Delta_E \mathcal{J}) + \dots$$

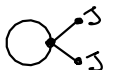
hence

$$W_{(1)}[\mathcal{J}] = -\frac{g}{4} \text{ (circle with two external legs)} + \frac{g^2}{16} \text{ (circle with four external legs)} + \dots$$

and by using the integral equation for $\phi_c[\mathcal{S}]$ we find graphs of the form



in which a single loop is decorated by trees. The lowest order perturbative

contribution is , from which we get the $O(\hbar)$ contribution

$$\text{to } W_{(1)} : -\frac{3}{4} \int d^D x_1 d^D y_1 d^D y_2 \Delta_E(x_1, x_1) \Delta_E(x_1, y_1) \Delta_E(x_1, y_2) \mathcal{J}(y_1) \mathcal{J}(y_2).$$

Up to $O(\hbar)$ we thus have

$$W[\mathcal{S}] = W_{(0)}[\mathcal{S}] + \hbar W_{(1)}[\mathcal{S}] = -S[\phi_c] + S\phi_c + \hbar W_{(1)}[\phi_c[\mathcal{S}]], \quad \text{where } \frac{\delta S}{\delta \phi}[\phi_c[\mathcal{S}]] = \mathcal{S}$$

$$\varphi = \frac{\delta W}{\delta \mathcal{S}} = -\frac{\delta S}{\delta \phi}[\phi_c] \frac{\delta \phi_c}{\delta \mathcal{S}} + \phi_c + \mathcal{S} \frac{\delta \phi_c}{\delta \mathcal{S}} + \hbar \frac{\delta W_{(1)}}{\delta \phi}[\phi_c] \frac{\delta \phi_c}{\delta \mathcal{S}}$$

$$= \phi_c + \hbar \frac{\delta W_{(1)}}{\delta \phi}[\phi_c] \frac{\delta \phi_c}{\delta \mathcal{S}} \quad \text{and } \frac{\delta \phi_c}{\delta \mathcal{S}} \text{ can be computed from } \frac{\delta^2 S}{\delta \phi \delta \phi}[\phi_c]$$

$W[\mathcal{S}] = \mathcal{S}\varphi - \Gamma[\varphi]$, and since $\Gamma[\varphi] - \mathcal{S}\varphi$ is stationary in φ (remember that $\frac{\delta \Gamma}{\delta \varphi} = \mathcal{S}$) we can use $\varphi \simeq \phi_c$ to compute $\Gamma[\varphi]$ up to $O(\hbar)$:

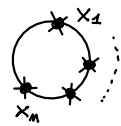
$$\Gamma[\varphi] = \mathcal{S}\varphi - (-S[\varphi] + \mathcal{S}\varphi + \hbar W_{(1)}[\varphi]) = S[\varphi] - \hbar W_{(1)}[\varphi] = \Gamma_{(0)}[\varphi] - \hbar W_{(1)}[\varphi]$$

$$\text{hence } \Gamma_{(1)}[\varphi] = -W_{(1)}[\varphi] = \frac{1}{2} \tau_2 \left[\log \frac{\delta^2 S}{\delta \varphi \delta \varphi} - \log \frac{\delta^2 S}{\delta \varphi \delta \varphi} \Big|_{\varphi=0} \right].$$

Proceeding as before we find

$$\Gamma_{(1)}[\varphi] = \frac{1}{2} \tau_2 \log \left(\delta(x_1 - x_2) + V''(\varphi) \Delta_E(x_1, x_2) \right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int d^D x_1 \dots d^D x_n V''(\varphi(x_1)) \Delta_E(x_1, x_2) \dots V''(\varphi(x_n)) \Delta_E(x_n, x_1)$$

where $V''(\varphi) = \frac{3}{2} \varphi^2$ for the ϕ^4 field theory.

This expression is the sum of diagrams of the form  where the vertex  denotes an insertion of $V''(\varphi)$ (note: 1PI diagrams)

For the effective potential $V_{\text{eff}}(\varphi) = \Gamma[\varphi] \Big|_{\varphi \text{ const.}}$ we have, up to $O(\hbar)$

$$V_{\text{eff}}(\varphi) = \int d^D x \left[\frac{1}{2} m^2 \varphi^2 + \frac{3}{4!} \varphi^4 \right] + \frac{1}{2} \hbar \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{3^n}{2^n} \varphi^{2n} \int d^D x_1 \dots d^D x_n \Delta_E(x_1, x_2) \dots \Delta_E(x_n, x_1)$$

By using $\Delta_E(x_1, x_2) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik(x_1 - x_2)}}{k^2 + m^2}$ we have

$$\begin{aligned} \int \Delta_E(x_1, x_2) \Delta_E(x_2, x_3) d^D x_2 &= \int d^D x_2 \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{e^{ik_1(x_1 - x_2)} e^{ik_2(x_2 - x_3)}}{(k_1^2 + m^2)(k_2^2 + m^2)} = \\ &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(2\pi)^D \delta(k_1 - k_2)}{(k_1^2 + m^2)(k_2^2 + m^2)} e^{i(k_1 x_1 - k_2 x_3)} = \int \frac{d^D k}{(2\pi)^D} \frac{e^{i(x_1 - x_3)k}}{(k^2 + m^2)^2} \end{aligned}$$

and we can rewrite ($V_D = \int d^D x$ = space-time volume)

$$V_{\text{eff}}(\varphi)/V_D = \frac{1}{2} m^2 \varphi^2 + \frac{3}{4!} \varphi^4 + \frac{1}{2} \hbar \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{3^n}{2^n} \varphi^{2n} I_n(m) \quad \text{where } I_n(m) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + m^2)^n}$$

NOTE: if $D=4$, both I_1 and I_2 are divergent.

We have seen that for the computation of $\Gamma_{(0)}$ we need tree diagrams, while 1-loop diagrams enter the computation of $\Gamma_{(1)}$. Let us show that The semiclassical expansion coincides with the loop expansion, hence $\Gamma_{(n)}$ correspond to n -loop (1PI amputated) diagrams

$Z[\mathcal{J}] \propto \int \mathcal{D}\phi e^{-\frac{1}{\hbar}(S-\mathcal{J}\phi)}$, with $\frac{S}{\hbar} = \frac{1}{2\hbar} \phi k \phi + \frac{1}{\hbar} V$, and $W[\mathcal{J}] = \hbar \log Z[\mathcal{J}]$, hence

propagator $\sim \hbar$, vertices $\sim 1/\hbar$, external source $\sim 1/\hbar$. A diagram contributing to

$W_n(x_1, \dots, x_n)$ thus has the power of $\hbar$ given by $\hbar^{I-V+1+n-m}$, where

$I = \# \text{ internal lines}$, $V = \# \text{ vertices}$, $+1$ is due to $W = \hbar \log Z$, $+n$ is due to the n external lines, and $-m$ is due to the external sources.

For plane polygons we have Euler relation: $L = I - V + 1$ where L is the number of faces, i.e.

the number of loops. To prove this relation one can use the fact that the relation is true

for a triangle and it is invariant if we add chords ($L \rightarrow L+1$, $I \rightarrow I+3$, $V \rightarrow V+2$) or

remove faces of a triangulated polygon ($L \rightarrow L-1$, $I \rightarrow I-2$, $V \rightarrow V-1$), hence it is true

for a generic polygon. From the point of view of electrical circuits L is the number of

loop currents, which is given by $I - V + 1$ since I is the total number of link currents, and

$V-1$ is the number of independent constraints due to current conservation at vertices ($V-1$ since the global current conservation does not fix a current).

NOTE: if $\frac{1}{\hbar}(S-\mathcal{J}\phi) = \frac{1}{2\hbar} \phi k \phi + \frac{g}{4! \hbar} \phi^4 - \frac{1}{\hbar} \mathcal{J}\phi$ and we perform the substitution $\phi \rightarrow \phi \sqrt{\hbar}$ we get

$\frac{1}{\hbar}(S-\mathcal{J}\phi) \rightarrow \frac{1}{2} \phi k \phi + \frac{g\hbar}{4!} \phi^4 - \frac{1}{\sqrt{\hbar}} \mathcal{J}\phi$, hence the power counting for $W_n(x_1, \dots, x_n)$ is

$g^V \hbar^{V+1-n/2}$ hence the loop expansion is equivalent (for any fixed n) to the perturbative exp.

If we consider $\Gamma(\varphi)$ as an action and we introduce the fictitious parameter ϵ and the generating functional $W_\Gamma[\mathcal{J}]$ by

$$\exp[W_\Gamma[\mathcal{J}]/\epsilon] = \frac{\int \mathcal{D}\varphi e^{-\frac{1}{\epsilon}(\Gamma[\varphi] - \mathcal{J}\varphi)}}{\int \mathcal{D}\varphi e^{-\Gamma[\varphi]/\epsilon}}$$

in the limit $\epsilon \rightarrow 0$ tree diagrams dominate, and the saddle point solution φ_c is determined

by the equation $\frac{\delta \Gamma}{\delta \varphi}[\varphi_c] = \mathcal{J}$. In this limit $-W_\Gamma^{(0)} = \Gamma(\varphi_c) - \mathcal{J}\varphi_c$ (see pag. PSF14 above), and

$\frac{\delta W_\Gamma^{(0)}}{\delta \mathcal{J}} = \varphi_c + \int \frac{\delta \varphi_c}{\delta \mathcal{J}} \frac{\delta}{\delta \varphi_c} (\Gamma[\varphi_c] - \mathcal{J}\varphi_c) = \varphi_c$ due to the saddle point equation, hence $W_\Gamma^{(0)}$ is

the Legendre transform of Γ , i.e. $W[\mathcal{J}]$. This shows that if we use Γ as action (thus propagators

corresponds to $W_2(x_1, x_2)$, the inverse of $\Gamma_2(x_1, x_2)$, and vertices to $-\Gamma_n$, with $n \geq 2$), we can write

W_n using only tree diagrams, and from this fact it follows that only 1PI diagrams contribute to

Γ_n with $n \geq 2$. This proof is the same of WEINBERG "The quantum theory of fields" § 16.1,

for a different proof see ZINN-SUSTIN "Quantum field theory and critical phenomena" § 7.8

COHERENT STATES PATH-INTEGRALS

A fundamental ingredient of the discussion in the previous section has been the possibility of using a basis $|\phi, x\rangle$ such that $\hat{\phi}(x)|\phi, x\rangle = \phi(x)|\phi, x\rangle$, and the existence of such a basis was ensured by the fact that $[\hat{\phi}(t, x), \hat{\phi}(t, y)] = 0$, hence all field operators could be diagonalized simultaneously. When using fermion fields we have $\{\hat{\psi}(t, x), \hat{\psi}(t, y)\} = 0$, and we can not assume anymore the existence of the diagonal basis. To overcome this obstacle we now introduce coherent states path-integration, which starts from the normal ordered second quantization

$$\text{Hamiltonian } H = \sum_{i,j} t_{ij} a_i^\dagger a_j + \sum_{i,j,l} U_{ijl} a_i^\dagger a_j^\dagger a_l a_l$$

COHERENT STATES (BOSONS)

We have $[a_i, a_j^\dagger] = \delta_{ij}$ and if we define, for $\phi_i \in \mathbb{C}$, $|\phi\rangle = e^{\sum \phi_i a_i^\dagger} |0\rangle$ (where $a_i |0\rangle = 0$)

we have (since $[a_i^\dagger, a_j^\dagger] = 0$ we consider for the sake of the simplicity just a single d.o.f.)

$$\begin{aligned} a |\phi\rangle &= a e^{\phi a^\dagger} |0\rangle = a \sum_{n=0}^{\infty} \frac{\phi^n}{n!} (a^\dagger)^n |0\rangle = \sum_{n=1}^{\infty} \frac{\phi^n}{n!} [a, (a^\dagger)^n] |0\rangle = [a, (a^\dagger)^n] = n (a^\dagger)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{\phi^n}{(n-1)!} (a^\dagger)^{n-1} |0\rangle = \phi \sum_{j=0}^{\infty} \frac{\phi^j}{j!} (a^\dagger)^j |0\rangle = \phi e^{\phi a^\dagger} |0\rangle = \phi |\phi\rangle \end{aligned}$$

hence $a_i |\phi\rangle = \phi_i |\phi\rangle$. This property defines coherent states (note: $\langle \phi | \phi \rangle \neq 1$, see below)

It is immediate to check that the operator $a^\dagger$ can not have eigenstates: if $a^\dagger |v\rangle = \lambda |v\rangle$ then $\lambda \sum_{n=0}^{\infty} c_n |n\rangle = a^\dagger \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n+1} |n+1\rangle \Rightarrow \lambda c_{n+1} = \sqrt{n+1} c_n$ and from $\sum_{n=0}^{\infty} |c_n|^2 < \infty$ it follows $c_n = 0$.

Fundamental properties of coherent states:

$$1) \langle 0 | \phi \rangle = \langle 0 | \sum_{n=0}^{\infty} \frac{\phi^n (a^\dagger)^n}{n!} |0\rangle = \langle 0 | 0 \rangle = 1 \quad \text{since } \langle 0 | a^\dagger = 0$$

$$2) \text{ from } a_i |\phi\rangle = \phi_i |\phi\rangle \text{ we also have } \langle \phi | a_i^\dagger = \langle \phi | \phi_i^*$$

$$3) a^\dagger |\phi\rangle = \sum_{n=0}^{\infty} a^\dagger \frac{\phi^n (a^\dagger)^n}{n!} |0\rangle = \sum_{j=1}^{\infty} j \frac{\phi^{j-1} (a^\dagger)^{j-1}}{(j-1)!} |0\rangle = \frac{\partial}{\partial \phi} \sum_{j=0}^{\infty} \frac{\phi^j (a^\dagger)^j}{j!} |0\rangle = \frac{\partial}{\partial \phi} |\phi\rangle$$

$$4) \langle 0 | \phi \rangle = \langle 0 | e^{\sum \phi_i^* a_i} | \phi \rangle = \langle 0 | e^{\sum \phi_i^* \phi_i} | \phi \rangle = e^{\sum \phi_i^* \phi_i} \langle 0 | \phi \rangle = e^{\sum \phi_i^* \phi_i}$$

$a_i |\phi\rangle = \phi_i |\phi\rangle$ $\langle 0 | \phi \rangle = 1$ see 1)

and in particular $\langle \phi | \phi \rangle = \exp \{ \sum \phi_i^* \phi_i \}$.

5) Completeness relation (often called overcompleteness since the states $|\phi\rangle$ are not orthogonal)

let us define $d\phi_i d\phi_i^* \equiv d\text{Re} \phi_i d\text{Im} \phi_i$, then $\int \prod_i \frac{d\phi_i d\phi_i^*}{\pi} e^{-\sum \phi_i^* \phi_i} |\phi\rangle \langle \phi| = 1$

Note that from $z=x+iy$, $z^*=x-iy$ it immediately follows that $dz dz^* = -2i dx dy$

and for this reason, depending on the definition of $d\phi_i d\phi_i^*$, the integration measure

$\frac{d\phi_i d\phi_i^*}{2\pi i}$ can be found, instead of $\frac{d\phi_i d\phi_i^*}{\pi}$, in the completeness relation.

To prove the completeness relation let us show that

$$\langle \Theta | \Psi \rangle = \int \prod_i \frac{d\phi_i d\phi_i^*}{\pi} e^{-\sum \phi_i^* \phi_i} \langle \Theta | \phi \rangle \langle \phi | \Psi \rangle$$

by using for the sake of the simplicity the case of a single degree of freedom. We start by

noting that $|\phi\rangle = e^{\phi a^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{\phi^n (a^\dagger)^n}{n!} |0\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}} |n\rangle$, where $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$ is a number state. Then

$$\int \frac{d\phi d\phi^*}{\pi} e^{-\phi^* \phi} \langle \Theta | \phi \rangle \langle \phi | \Psi \rangle = \int \frac{d\phi d\phi^*}{\pi} e^{-\phi^* \phi} \sum_{m,j=0}^{\infty} \frac{\phi^m}{\sqrt{m!}} \frac{(\phi^*)^j}{\sqrt{j!}} \langle \Theta | m \rangle \langle j | \Psi \rangle$$

and passing to polar coordinates $\phi = x e^{i\varphi}$, $d\phi d\phi^* = x dx d\varphi$ we have

$$\begin{aligned} & \int \frac{x dx d\varphi}{\pi} e^{-x^2} \sum_{m,j=0}^{\infty} \frac{e^{i(m-j)\varphi}}{\sqrt{m!} \sqrt{j!}} \langle \Theta | m \rangle \langle j | \Psi \rangle x^{m+j} = \int d\varphi e^{i(m-j)\varphi} = 2\pi \delta_{m,j} \\ & = \sum_{m=0}^{\infty} \int_0^{\infty} \frac{x dx}{m!} e^{-x^2} x^{2m} \langle \Theta | m \rangle \langle m | \Psi \rangle = \sum_{m=0}^{\infty} \frac{\langle \Theta | m \rangle \langle m | \Psi \rangle}{m!} \int_0^{\infty} t^m e^{-t} dt = \sum_{m=0}^{\infty} \langle \Theta | m \rangle \langle m | \Psi \rangle = \langle \Theta | \Psi \rangle \end{aligned}$$

$x^2 = t$ $\Gamma(m+1) = m!$

For some more formal details see KLAUDER, SUDARSHAN "Fundamentals of Quantum Optics" § 7.

COHERENT STATES (FERMIONS)

From $\{a_i, a_j\} = 0$ it easily follows that eigenvalues of the operator a_i can not be simple

complex numbers: from $a_i |\eta\rangle = \eta_i |\eta\rangle$ we have

$$\begin{aligned} a_j a_i |\eta\rangle &= a_j \eta_i |\eta\rangle = \eta_i a_j |\eta\rangle = \eta_i \eta_j |\eta\rangle \\ &\stackrel{!}{=} -a_i a_j |\eta\rangle = -a_i \eta_j |\eta\rangle = -\eta_j a_i |\eta\rangle = -\eta_j \eta_i |\eta\rangle \end{aligned}$$

We thus introduce GRASSMAN variables, i.e. variables which satisfy the relations

$$\{\eta_i, \eta_j\} = 0, \quad \{\eta_i, a_j\} = 0, \quad \{\eta_i, a_j^\dagger\} = 0, \quad [\eta_i, \rho] = 0 \text{ if } \rho \in \mathbb{C}$$

Functions of Grassmann variables are defined by their Taylor expansions

$$1 \text{ d.o.f. } F(\eta) = A + B\eta, \quad A, B \in \mathbb{C}$$

$$2 \text{ d.o.f. } F(\eta) = A + B\eta_1 + C\eta_2 + D\eta_1\eta_2, \quad A, B, C, D \in \mathbb{C}$$

and a function of N variables is, in the most general case, a polynomial of degree N .

Let us now define $|\eta\rangle = e^{-\sum \eta_i a_i^\dagger} |0\rangle = \prod_i e^{-\eta_i a_i^\dagger} |0\rangle$, where the second equality follows

from the fact that $[\eta_i a_i^\dagger, \eta_j a_j^\dagger] = 0$: products of two anticommuting variables are commuting variables.

With this definition we have

$$a_i |\eta\rangle = a_i \prod_j e^{-\eta_j a_j^\dagger} |0\rangle = \left(\prod_{j \neq i} e^{-\eta_j a_j^\dagger} \right) a_i e^{-\eta_i a_i^\dagger} |0\rangle$$

$$\uparrow [a_i, \eta_j a_j^\dagger] = 0 \text{ if } i \neq j$$

$$\{ \eta_i, \eta_i \} = 0$$

$$a_i e^{-\eta_i a_i^\dagger} |0\rangle = a_i (1 - \eta_i a_i^\dagger) |0\rangle = \eta_i a_i a_i^\dagger |0\rangle = \eta_i |0\rangle = \eta_i (1 - \eta_i a_i^\dagger) |0\rangle = \eta_i e^{-\eta_i a_i^\dagger} |0\rangle$$

$$\uparrow \{ a_i, \eta_i \} = 0$$

$$\text{hence } a_i |\eta\rangle = \left(\prod_{j \neq i} e^{-\eta_j a_j^\dagger} \right) \eta_i e^{-\eta_i a_i^\dagger} |0\rangle = \eta_i e^{-\sum_j \eta_j a_j^\dagger} |0\rangle = \eta_i |\eta\rangle$$

$$\uparrow [\eta_i, \eta_j a_j^\dagger] = 0$$

The relation $a_i |\eta\rangle = \eta_i |\eta\rangle$ defines fermionic coherent states. It is also simple to compute $\langle 0 | \eta \rangle$:

$$\langle 0 | \eta \rangle = \langle 0 | \prod_i (1 - \eta_i a_i^\dagger) |0\rangle = \langle 0 | \prod_i (1 + a_i^\dagger \eta_i) |0\rangle = \langle 0 | 0 \rangle = 1, \text{ as in the bosonic case.}$$

$$\uparrow \{ a_i^\dagger, \eta_i \} = 0$$

$$\uparrow \langle 0 | a_i^\dagger = 0$$

Let us now define derivatives with respect to Grassmann variables by the rule $\frac{\partial}{\partial \eta_i} (\eta_i F) = F$, where it is important that in $\eta_i F$ the variable η_i appears first, and F is independent of F (otherwise everything vanishes). From

$$\frac{\partial}{\partial \eta_1} \frac{\partial}{\partial \eta_2} (\eta_2 \eta_1) = \frac{\partial}{\partial \eta_1} \eta_1 = 1 = \frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_1} (\eta_1 \eta_2) = - \frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_1} (\eta_2 \eta_1) \quad \text{we have } \left\{ \frac{\partial}{\partial \eta_i}, \frac{\partial}{\partial \eta_j} \right\} = 0$$

and for consistency with $\{ \eta_i, a_j \} = \{ \eta_i, a_j^\dagger \} = 0$ we also need $\left\{ \frac{\partial}{\partial \eta_i}, a_j \right\} = \left\{ \frac{\partial}{\partial \eta_i}, a_j^\dagger \right\} = 0$.

We can verify that $\{ \eta_i, \frac{\partial}{\partial \eta_j} \} = \delta_{ij}$, indeed if $i \neq j$ we have $F = A \eta_i \eta_j + B \eta_i + C \eta_j + D$

$$\eta_i \frac{\partial}{\partial \eta_j} F + \frac{\partial}{\partial \eta_j} (\eta_i F) = \eta_i (-A \eta_i + C) + \frac{\partial}{\partial \eta_j} (C \eta_i \eta_j + D \eta_i) = C \eta_i - C \eta_i = 0$$

while for $i=j$ we have $F = A \eta + B$ and $\eta \frac{\partial}{\partial \eta} F + \frac{\partial}{\partial \eta} (\eta F) = A \eta + \frac{\partial}{\partial \eta} (B \eta) = A \eta + B = F$

By an analogous direct computation it is possible to show the validity of the chain rule

$$\frac{\partial}{\partial \theta} F(\eta, \psi) = \frac{\partial \eta}{\partial \theta} \frac{\partial F}{\partial \eta} + \frac{\partial \psi}{\partial \theta} \frac{\partial F}{\partial \psi}$$

where the ordering of the various terms can not be changed.

$$a_i^\dagger |\eta\rangle = a_i^\dagger e^{-\sum_j \eta_j a_j^\dagger} |0\rangle = a_i^\dagger \prod_j (1 - \eta_j a_j^\dagger) |0\rangle = a_i^\dagger \prod_{j \neq i} (1 - \eta_j a_j^\dagger) |0\rangle =$$

$$\uparrow (a_j^\dagger)^2 = 0$$

$$= - \frac{\partial}{\partial \eta_i} \prod_j (1 - \eta_j a_j^\dagger) |0\rangle = - \frac{\partial}{\partial \eta_i} \exp(-\sum_j \eta_j a_j^\dagger) |0\rangle = - \frac{\partial}{\partial \eta_i} |\eta\rangle$$

Let us now define $\langle \eta | = \langle 0 | e^{-\sum_i a_i \bar{\eta}_i}$ where $\bar{\eta}_i$ are new Grassmann numbers, then

$$\langle \eta | a_i^\dagger = \langle 0 | \exp(-\sum_j a_j \bar{\eta}_j) a_i^\dagger = \langle 0 | \prod_j (1 - a_j \bar{\eta}_j) a_i^\dagger = \langle 0 | a_i a_i^\dagger \bar{\eta}_i \prod_{j \neq i} (1 - a_j \bar{\eta}_j) =$$

$$= \langle 0 | \bar{\eta}_i \prod_{j \neq i} (1 - a_j \bar{\eta}_j) = \langle 0 | \bar{\eta}_i (1 - a_i \bar{\eta}_i) \prod_{j \neq i} (1 - a_j \bar{\eta}_j) = \langle 0 | \bar{\eta}_i e^{-\sum_j a_j \bar{\eta}_j} = \langle 0 | e^{-\sum_j a_j \bar{\eta}_j} \eta_j = \langle \eta | \eta_j$$

hence $\langle \eta | a_i^\dagger = \langle \eta | \bar{\eta}_i$, which is the analogous of $\langle \phi | a_i^\dagger = \langle \phi | \phi_i^*$ for the bosonic case (which follows from $a_i |\phi\rangle = \phi_i |\phi\rangle$ by hermitian conjugation).

$\bar{\eta}_i$ is a sort of complex conjugate of η_i and we impose $\{\eta_i, \bar{\eta}_j\} = 0$, in such a way that $\sum \bar{\eta}_i \eta_i$ is a commuting variable (although not a $\mathbb{C}$ variable). In this way we have

$$\begin{aligned} \langle \eta | \theta \rangle &= \langle \eta | e^{-\sum \theta_i a_i^\dagger} | 0 \rangle = \langle \eta | \prod_i (1 - \theta_i a_i^\dagger) | 0 \rangle = \langle \eta | \prod_i (1 + a_i^\dagger \theta_i) | 0 \rangle = \langle \eta | \prod_i (1 + \bar{\eta}_i \theta_i) | 0 \rangle = \\ &= \langle \eta | e^{\sum \bar{\eta}_i \theta_i} | 0 \rangle = \exp(\sum \bar{\eta}_i \theta_i) \langle \eta | 0 \rangle = \exp(\sum \bar{\eta}_i \theta_i) \quad \text{since it is easy to see that } \langle \eta | 0 \rangle = 1. \end{aligned}$$

Note that $\langle \eta | \theta \rangle$ is not a complex number, however it commutes with all Grassman variables.

It is not difficult to show that $\langle \eta | a_i = \frac{\partial}{\partial \bar{\eta}_i} \langle \eta |$

Let us now introduce the integration rules for Grassman variables: integral = derivative, i.e.

$$\int d\eta_i F(\eta) = \frac{\partial}{\partial \eta_i} F(\eta)$$

hence we have in particular $\{d\eta_i, d\eta_j\} = 0$ and $\int d\eta (A+B\eta) = B$. The integral defined in this way is automatically invariant under translations and if $\eta = C\eta'$ we have

$$A+B\eta = A+BC\eta' \quad \text{and} \quad \int d\eta' (A+BC'\eta') = BC$$

hence for consistency we need $d\eta = d\eta' / C$, which is the inverse of the usual transformation rule for differentials. In general $\int d\eta_1 \dots d\eta_n f =$ coefficient of $\eta_n \dots \eta_1$ in the Taylor exp of f , and if we perform the change of variable $\eta_i \rightarrow \eta'_i = \sum_j A_{ij} \eta_j$

$$\eta'_1 \dots \eta'_n = \sum_{j_1 \dots j_n} A_{1j_1} \eta_{j_1} \dots A_{nj_n} \eta_{j_n} = \eta_1 \dots \eta_n \sum_{\text{perm. of } 1 \dots n} (-1)^\sigma A_{1j_1} \dots A_{nj_n} = \eta_1 \dots \eta_n \det A$$

hence for consistency we need $d\eta_1 \dots d\eta_n = \frac{1}{\det A} d\eta'_1 \dots d\eta'_n$, again the inverse of what happens in the standard $\mathbb{R}^n$ case.

Complex Gaussian integral: Let A be a $N \times N$ hermitian matrix with positive eigenvalues.

$$I = \int \prod_i d\bar{z}_i dz_i^* \exp\left(-\sum_{ij} \bar{z}_i^* A_{ij} z_j\right) \quad \text{where } d\bar{z}_i dz_i^* = d\text{Re} z_i d\text{Im} z_i \text{ as before}$$

A unitary matrix U exists such that $A = U^\dagger \Lambda U$, with $\Lambda =$ diagonal matrix with real and positive diagonal elements. By defining $z_j = \sum_k U_{jk} y_k$ and $\bar{z}_i^* = \sum_l U_{il}^* y_l^*$ we have

$\sum_{ij} \bar{z}_i^* A_{ij} z_j = \sum_l \lambda_l |y_l|^2$ moreover $\prod_i d\bar{z}_i dz_i^*$ is invariant since $|\det U|^2 = 1$, hence we get

$$I = \left(\int \prod_i dy_i e^{-\sum_l \lambda_l y_l^2} \right)^2 = \left(\prod_i \sqrt{\frac{\pi}{\lambda_i}} \right)^2 = \frac{\pi^N}{\det A}$$

So if we define

$$Z(A, b, b^*) = \int \prod_i \frac{d\bar{z}_i dz_i^*}{\pi} \exp \left\{ -\sum_{ij} \bar{z}_i^* A_{ij} z_j + \sum_i (b_i^* \bar{z}_i + b_i z_i^*) \right\} \quad A = A^\dagger \Rightarrow (A^{-1})^\dagger = A^{-1}$$

using the change of variable $z_i = y_i + \sum_j (A^{-1})_{ij} b_j$; $\bar{z}_i^* = y_i^* + \sum_j (A^{-1})_{ji} b_j^*$

it is immediate to get $z(A, b, b^*) = \frac{\pi^N}{\det A} \exp \left\{ + \sum_{i,j} b_i^* (A^{-1})_{ij} b_j \right\}$

Let us now study Gaussian integrals for Grassmann variables. We start from

$$z(A) = \int d\eta_1 d\bar{\eta}_1 \dots d\eta_m d\bar{\eta}_m \exp \left(\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j \right)$$

where care is needed to keep constant the ordering of the differentials, since $\{d\eta_i, d\eta_j\} = 0$ and $\{d\eta_i, d\bar{\eta}_j\} = 0$, $\{d\bar{\eta}_i, d\bar{\eta}_j\} = 0$.

$$\exp \left(\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j \right) = \prod_{i=1}^m \exp \left(\bar{\eta}_i \sum_j A_{ij} \eta_j \right) = \prod_{i=1}^m (1 + \bar{\eta}_i \sum_j A_{ij} \eta_j)$$

hence

$$z(A) = \int d\eta_1 d\bar{\eta}_1 \dots d\eta_m d\bar{\eta}_m \prod_{i=1}^m \left(\bar{\eta}_i \sum_j A_{ij} \eta_j \right) = \int d\eta_1 d\bar{\eta}_1 \dots d\eta_m d\bar{\eta}_m \sum_{\text{perm}} A_{m j_m} \dots A_{1 j_1} \bar{\eta}_m \eta_{j_m} \dots \bar{\eta}_1 \eta_{j_1}$$

Since by exchanging the positions of two η 's we get a minus sign, the coeff. of $\bar{\eta}_m \eta_{j_m} \dots \bar{\eta}_1 \eta_{j_1}$ is

$$z(A) = \sum_{\text{perm}} (-1)^{\sigma} A_{m j_m} \dots A_{1 j_1} = \det A$$

which is, apart from some numerical factors, the inverse of the result that is obtained in the complex case. Note also that

$$\int d\eta_1 d\bar{\eta}_1 \dots d\eta_m d\bar{\eta}_m \exp \left(\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j \right) = \int d\bar{\eta}_1 d\eta_1 \dots d\bar{\eta}_m d\eta_m \exp \left(- \sum_{i,j} \bar{\eta}_i A_{ij} \eta_j \right)$$

since the different sign in the exponential gives a $(-1)^m$ which cancels the $(-1)^m$ that is generated by switching the differentials: $d\eta_i d\bar{\eta}_i = -d\bar{\eta}_i d\eta_i$.

$$\text{If we now consider } z(A, \theta, \bar{\theta}) = \int \prod_i d\bar{\eta}_i d\eta_i \exp \left\{ - \sum_{i,j} \bar{\eta}_i A_{ij} \eta_j + \sum_i (\bar{\theta}_i \eta_i + \bar{\eta}_i \theta_i) \right\},$$

with the change of variables

$$\eta_i = \eta_i' + \sum_j (A^{-1})_{ij} \theta_j \quad \text{and} \quad \bar{\eta}_i = \bar{\eta}_i' + \sum_j \bar{\theta}_j (A^{-1})_{ji}$$

we get easily

$$z(A, \theta, \bar{\theta}) = (\det A) \exp \left\{ \sum_{i,j} \bar{\theta}_i (A^{-1})_{ij} \theta_j \right\}$$

It is easy to see that

$$\frac{1}{\det A} \frac{\partial}{\partial \theta_j} z(A, \theta, \bar{\theta}) = - \left(\sum_i \bar{\theta}_i (A^{-1})_{ij} \right) e^{\sum_{i,k} \bar{\theta}_i (A^{-1})_{ik} \theta_k}; \quad \frac{1}{\det A} \frac{\partial}{\partial \bar{\theta}_i} \frac{\partial}{\partial \theta_j} z(A, \theta, \bar{\theta}) \Big|_{\theta=\bar{\theta}=0} = - (A^{-1})_{ij}$$

Moreover, from the definition of z we have

$$\frac{\partial}{\partial \theta_j} z = \int \prod_k d\bar{\eta}_k d\eta_k (-\bar{\eta}_j) \exp \left\{ - \sum_{i,j} \bar{\eta}_i A_{ij} \eta_j + \sum_i (\bar{\theta}_i \eta_i + \bar{\eta}_i \theta_i) \right\},$$

$$\frac{\partial}{\partial \bar{\theta}_i} \frac{\partial}{\partial \theta_j} z = \int \prod_k d\bar{\eta}_k d\eta_k (\bar{\eta}_j \eta_i) \exp \left\{ - \sum_{i,j} \bar{\eta}_i A_{ij} \eta_j + \sum_i (\bar{\theta}_i \eta_i + \bar{\eta}_i \theta_i) \right\}$$

hence

$$\langle \bar{\eta}_j \eta_i \rangle = \frac{\int \prod_k d\bar{\eta}_k d\eta_k (\bar{\eta}_j \eta_i) \exp\left\{-\sum_{ij} \bar{\eta}_i A_{ij} \eta_j\right\}}{\int \prod_k d\bar{\eta}_k d\eta_k \exp\left\{-\sum_{ij} \bar{\eta}_i A_{ij} \eta_j\right\}} = \frac{1}{\det A} \frac{\partial}{\partial \bar{\theta}_i} \frac{\partial}{\partial \theta_j} \mathcal{Z}(A, \theta, \bar{\theta}) \Big|_{\theta=\bar{\theta}=0} = -(A^{-1})_{ij}$$

and $\langle \eta_i \bar{\eta}_j \rangle = (A^{-1})_{ij}$. For 2n-point functions we have

$$\begin{aligned} \langle \bar{\eta}_{i_1} \eta_{j_1} \dots \bar{\eta}_{i_m} \eta_{j_m} \rangle &= \frac{1}{\det A} \frac{\partial}{\partial \bar{\theta}_{j_1}} \frac{\partial}{\partial \theta_{i_1}} \dots \frac{\partial}{\partial \bar{\theta}_{j_m}} \frac{\partial}{\partial \theta_{i_m}} \mathcal{Z}(A, \theta, \bar{\theta}) \Big|_{\theta=\bar{\theta}=0} = \\ &= \frac{1}{\det A} \int d\bar{\theta}_{j_1} d\theta_{i_1} \dots d\bar{\theta}_{j_m} d\theta_{i_m} \det A e^{\sum \bar{\theta}_\alpha A_{\alpha\beta} \theta_\beta} \Big|_{\theta=\bar{\theta}=0} = \det [(-A^{-1})_{j_i k}] = \\ &= \det [\langle \bar{\eta}_{i_k} \eta_{j_l} \rangle] = \sum_{\substack{\text{perm.} \\ \text{di } j_1 \dots j_m}} (-1)^\sigma \langle \bar{\eta}_{i_1} \eta_{j_{p_1}} \rangle \dots \langle \bar{\eta}_{i_m} \eta_{j_{p_m}} \rangle \quad \text{Wick theorem for fermions} \end{aligned}$$

We can now prove the (over)completeness for fermionic coherent states

$$\int d\bar{\eta}_1 d\eta_1 \dots d\bar{\eta}_m d\eta_m e^{-\sum \bar{\eta}_i \eta_i} |\eta\rangle \langle \eta| = 1$$

We thus have to show that

$$\int d\bar{\eta}_1 d\eta_1 \dots d\bar{\eta}_m d\eta_m e^{-\sum \bar{\eta}_i \eta_i} \langle \psi | \eta \rangle \langle \eta | \theta \rangle = \langle \psi | \theta \rangle$$

but we have $\langle \psi | \eta \rangle = e^{\sum \bar{\psi}_i \eta_i}$, $\langle \eta | \theta \rangle = e^{\sum \bar{\eta}_i \theta_i}$, and $\langle \psi | \theta \rangle = e^{\sum \bar{\psi}_i \theta_i}$. Using the expression for Gaussian integrals (with $A=I$) we have

$$\int d\bar{\eta}_1 d\eta_1 \dots d\bar{\eta}_m d\eta_m \exp\left(-\sum_i \bar{\eta}_i \eta_i + \sum_i \bar{\psi}_i \eta_i + \sum_i \bar{\eta}_i \theta_i\right) = e^{\sum \bar{\psi}_i \theta_i} = \langle \psi | \theta \rangle$$

which completes the proof.

COHERENT STATES PATH-INTEGRAL

Before introducing path-integrals in the coherent state basis we have to discuss how to write

traces in this basis. Let us start from the bosonic case; using the completeness relation we have

$$\begin{aligned} \text{tr}(A) &= \sum_n \langle n | A | n \rangle = \sum_n \int \prod_i \frac{d\phi_i d\phi_i^*}{\pi} e^{-\sum \phi_i^* \phi_i} \langle n | \phi \rangle \langle \phi | A | n \rangle = \\ &= \sum_n \int \prod_i \frac{d\phi_i d\phi_i^*}{\pi} e^{-\sum \phi_i^* \phi_i} \langle \phi | A | n \rangle \langle n | \phi \rangle = \int \prod_i \frac{d\phi_i d\phi_i^*}{\pi} e^{-\sum \phi_i^* \phi_i} \langle \phi | A | \phi \rangle \end{aligned}$$

For the fermionic case we have to pay attention to the anticommuting nature of Grassmann variables

$$\text{tr}(A) = \sum_k \langle k | A | k \rangle = \sum_k \underbrace{\int \prod_i d\bar{\eta}_i d\eta_i e^{-\sum \bar{\eta}_i \eta_i}}_{\text{even \# of variables}} \langle k | \eta \rangle \langle \eta | A | k \rangle$$

and it is not difficult to verify that if $|k\rangle$ is a number state we have $\langle k | \eta \rangle \langle \eta | k \rangle = \langle -\eta | k \rangle \langle k | \eta \rangle$;

from $|k\rangle = a_1^\dagger \dots a_k^\dagger |0\rangle$ we have $\langle k | \eta \rangle = \langle 0 | a_k \dots a_1 | \eta \rangle = \eta_k \dots \eta_1$ and analogously we have

$$\langle \eta | k \rangle = \bar{\eta}_1 \dots \bar{\eta}_k, \text{ hence}$$

$$\begin{aligned}\langle k|\eta\rangle\langle\eta|k\rangle &= \eta_k\cdots\eta_1\bar{\eta}_1\cdots\bar{\eta}_k = \eta_1\bar{\eta}_1\cdots\eta_k\bar{\eta}_k = (-\bar{\eta}_1\eta_1)\cdots(-\bar{\eta}_k\eta_k) = \\ &= (-\bar{\eta}_1)\cdots(-\bar{\eta}_k)\eta_k\cdots\eta_1 = \langle\eta|k\rangle\langle k|\eta\rangle.\end{aligned}$$

For the trace we thus have

$$\text{tr}(A) = \sum_k \int \prod_i d\bar{\eta}_i d\eta_i e^{-\sum_i \bar{\eta}_i \eta_i} \langle -\eta|A|k\rangle \langle k|\eta\rangle = \int \prod_i d\bar{\eta}_i d\eta_i e^{-\sum_i \bar{\eta}_i \eta_i} \langle -\eta|A|\eta\rangle.$$

We can now use these expressions for the trace, and the completeness relation, to write the path-integral in the coherent basis, starting from the Hamiltonian written in second quantization:

$$\hat{H} = \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} U_{ijkl} a_i^\dagger a_j^\dagger a_k a_l$$

Note that by a redefinition of the numbers h_{ij} we can also take into account a chemical potential

$\hat{H} \rightarrow \hat{H} - \mu \hat{N}$ by $h_{ij} \rightarrow h_{ij} - \mu \delta_{ij}$. It is important to note that $\hat{H}$ is normal ordered, i.e.

all creation operators are on the left of the annihilation operators; this will enable us to diagonalize

$\hat{H}$ using the coherent state basis.

Let us start from the bosonic case: we have seen that (we add a subscript 0 to integration variables for later convenience)

$$\mathcal{Z} = \text{tr}(e^{-\beta\hat{H}}) = \int \prod_i \frac{d\phi_{0,i} d\phi_{0,i}^*}{\pi} \langle \phi_0 | e^{-\beta\hat{H}} | \phi_0 \rangle e^{-\sum_i \phi_{0,i}^* \phi_{0,i}}$$

then we write

$$\langle \phi_0 | e^{-\beta\hat{H}} | \phi_0 \rangle = \langle \phi_0 | e^{-\beta\hat{H}/N} \cdots e^{-\beta\hat{H}/N} | \phi_0 \rangle \text{ and we insert } N-1 \text{ completeness relations:}$$

$$\begin{aligned}\mathcal{Z} &= \int \prod_{m=0}^{N-1} \prod_i \frac{d\phi_{m,i} d\phi_{m,i}^*}{\pi} \langle \phi_0 | e^{-\beta\hat{H}/N} | \phi_{N-1} \rangle e^{-\sum_i \phi_{N-1,i}^* \phi_{N-1,i}} \langle \phi_{N-1} | e^{-\beta\hat{H}/N} | \phi_{N-2} \rangle \cdots \\ &\quad \cdots e^{-\sum_i \phi_{1,i}^* \phi_{1,i}} \langle \phi_1 | e^{-\beta\hat{H}/N} | \phi_0 \rangle e^{-\sum_i \phi_{0,i}^* \phi_{0,i}}\end{aligned}$$

For large N we have

$$\begin{aligned}\langle \phi_{m+1} | e^{-\beta\hat{H}/N} | \phi_m \rangle &\simeq \langle \phi_{m+1} | \left(1 - \frac{\beta}{N} \left\{ \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} U_{ijkl} a_i^\dagger a_j^\dagger a_k a_l \right\} \right) | \phi_m \rangle = \\ &= \left(1 - \frac{\beta}{N} \left\{ \sum_{ij} h_{ij} \phi_{m+1,i}^* \phi_{m,j} + \sum_{ijkl} U_{ijkl} \phi_{m+1,i}^* \phi_{m+1,j}^* \phi_{m,k} \phi_{m,l} \right\} \right) \langle \phi_{m+1} | \phi_m \rangle \simeq \\ &\simeq \exp\left(-\frac{\beta}{N} H(\phi_{m+1}^*, \phi_m)\right) e^{\sum_i \phi_{m+1,i}^* \phi_{m,i}}\end{aligned}$$

where we defined $H(\phi^*, \phi) = \frac{\langle \phi | H(a^\dagger, a) | \phi \rangle}{\langle \phi | \phi \rangle}$. We thus have

$$\mathcal{Z} = \int \prod_{m=0}^{N-1} \prod_i \frac{d\phi_{m,i} d\phi_{m,i}^*}{\pi} \exp\left\{-\frac{\beta}{N} \sum_{m=0}^{N-1} \left[\sum_i \left(\frac{\phi_{m,i}^* - \phi_{m+1,i}^*}{\beta/N} \right) \phi_{m,i} + H(\phi_{m+1}^*, \phi_m) \right] \right\}$$

$$\phi_0 = \phi_N$$

$$\phi_0^* = \phi_N^*$$

$$\text{moreover } \sum_{m,i} \phi_{m,i}^* \phi_{m,i} = \sum \phi_{m+1,i}^* \phi_{m+1,i} \text{ hence } \sum_{m,i} (\phi_{m,i}^* - \phi_{m+1,i}^*) \phi_{m,i} = \sum \phi_{m+1,i}^* (\phi_{m+1,i} - \phi_{m,i})$$

and in the limit $\beta/N \rightarrow 0$ we get

$$Z = \int_{\phi(0)=\phi(\beta)} \mathcal{D}(\phi^*, \phi) e^{-S[\phi^*, \phi]} \quad \text{with} \quad S[\phi^*, \phi] = \int_0^\beta dt (\phi^* \partial_t \phi + H(\phi^*, \phi))$$

which is the expression of the path-integral in the coherent basis (in a lattice model, like e.g. the Bose-Hubbard model, the sum on "i" remains discrete, since i labels the lattice sites).

If we introduce variables in the frequency domain by $\phi(t) = \frac{1}{\sqrt{\beta}} \sum_m \tilde{\phi}_m e^{-i\omega_m t}$ then the periodicity condition fixes $\omega_m = \frac{2\pi m}{\beta}$ (bosonic HATSUBARA frequencies) and the inverse transformation is

$$\tilde{\phi}_m = \frac{1}{\sqrt{\beta}} \int \phi(t) e^{i\omega_m t} dt. \quad \text{If we consider a quadratic Hamiltonian } H = \sum_{i,j} (h_{ij} - \mu \delta_{ij}) a_i^\dagger a_j$$

the action can be rewritten as

$$S = \int_0^\beta dt [\phi^* \partial_t \phi + H(\phi^*, \phi)] = \sum_{m,i,j} \tilde{\phi}_{m,i}^* (-i\omega_m - \mu) \delta_{ij} + h_{ij} \tilde{\phi}_{m,j}$$

For the fermionic case only the first step (i.e. the way in which the trace is written) is different

$$\text{and we find} \quad Z = \int \mathcal{D}(\bar{\eta}, \eta) e^{-S[\bar{\eta}, \eta]}$$

$$\eta(0) = -\eta(\beta)$$

$$\bar{\eta}(0) = -\bar{\eta}(\beta)$$

$$\text{where the action is} \quad S[\bar{\eta}, \eta] = \int_0^\beta (\bar{\eta} \partial_t \eta + H(\bar{\eta}, \eta)) dt =$$

$$= \int_0^\beta \left\{ \sum_{i,j} \bar{\eta}_i(t) [(\partial_t - \mu) \delta_{ij} + h_{ij}] \eta_j(t) + \sum_{ijkl} U_{ijkl} \bar{\eta}_i(t) \bar{\eta}_j(t) \eta_k(t) \eta_l(t) \right\} dt$$

Since fermionic fields are antiperiodic in time, in Fourier space the fermionic HATSUBARA frequencies are $\omega_m = \frac{2\pi m + 1}{\beta} \pi$.

For a bosonic perfect gas we have $S = \sum_{m,i,j} \tilde{\phi}_{m,i}^* [(-i\omega_m - \mu) \delta_{ij} + h_{ij}] \tilde{\phi}_{m,j}$ and we can perform a unitary (hence no change of the integration measure) transformation to diagonalize h_{ij} . If we denote by ϵ_i the single particle energies and define $\xi_i = \epsilon_i - \mu$, then

$$S = \sum_{m,i} \tilde{\phi}_{m,i}^* (-i\omega_m + \xi_i) \tilde{\phi}_{m,i} \quad \text{and the partition function is}$$

$$Z = \int \prod_{m,a} \frac{d\tilde{\phi}_{m,a}}{\sqrt{\pi}} \exp \left\{ - \sum_{m,a} \tilde{\phi}_{m,a}^* (-i\omega_m + \xi_a) \tilde{\phi}_{m,a} \right\} = \prod_a Z_a \quad \leftarrow \text{single particle partition function}$$

$$Z_a = \prod_m [\beta (-i\omega_m + \xi_a)]^{-1}$$

this is useful to make Z_a (which is defined up to multiplicative constants) dimensionless

Analogously, in the fermionic case, one finds $Z_a = \prod_m [\beta (-i\omega_m + \xi_a)]$, hence by introducing

$$S = \sum_{\sigma=\pm 1} \text{bosons} \quad \text{we have}$$

the free energy $F = -T \log Z = \sum T \sum_{\omega} \log [\beta(-i\omega + \xi_a)]$, which is however formally divergent. With some manipulations it is possible to isolate the additive divergent term, obtaining

$$F = \sum T \sum_{\omega} \log(1 - \beta e^{-\beta \xi_a})$$

see, e.g.

KAPUTSA, GALE "Finite temperature field theory" § 2.3, 2.5 (quite ad-hoc procedure)

ALTLAND, SIMONS "Condensed matter field theory" § 4.2 (using complex integration)

FRADKIN "Quantum field theory. An integrated approach" § 8.8 (using GELFAND-YAGLOM).

Relation between canonical and coherent state path-integral in a simple (QM) case

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2, \quad a = \sqrt{\frac{m\omega}{2}} \left(x + \frac{i}{m\omega} p \right), \quad [a, a^\dagger] = 1, \quad H = \omega \left(a^\dagger a + \frac{1}{2} \right)$$

Using the coherent state basis we have (apart from a trivial multiplicative constant)

$$Z = \int \mathcal{D}(\phi^*, \phi) \exp \left\{ - \int_0^{\beta} (\phi^* \partial_t \phi + \omega \phi^* \phi) dt \right\}$$

and using the substitution $\phi(t) = \sqrt{\frac{m\omega}{2}} \left(q(t) + \frac{i}{m\omega} p(t) \right)$ we have

$$\begin{aligned} d\phi d\phi^* &= d\text{Re}\phi d\text{Im}\phi \propto dq dp & \omega \phi^* \phi &= \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 & \text{integrating by part} \\ \phi^* \partial_t \phi &= \frac{m\omega}{2} \left(q - \frac{i}{m\omega} p \right) \left(\dot{q} + \frac{i}{m\omega} \dot{p} \right) = \underbrace{\frac{m\omega}{2} q \dot{q} + \frac{1}{2m\omega} p \dot{p}}_{\text{total derivative: it gives no contribution due to p.b.c.}} - \frac{i}{2} p \dot{q} + \frac{i}{2} q \dot{p} = -i p \dot{q} \end{aligned}$$

$$\begin{aligned} \text{hence } Z &= \int \mathcal{D}p \mathcal{D}q \exp \left\{ - \int_0^{\beta} \left(-i p \dot{q} + \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 \right) dt \right\} = \text{the quadratic integral in } p \text{ can be explicitly performed} \\ &= \int \mathcal{D}q \exp \left\{ - \int_0^{\beta} \left(\frac{m}{2} \dot{q}^2 + \frac{1}{2} m \omega^2 q^2 \right) dt \right\} = \int \mathcal{D}q \exp \left\{ - \int_0^{\beta} \mathcal{L}_E dt \right\} \quad p \rightarrow +i m \dot{q} \end{aligned}$$

DIRAC FERMIONS

In the many-body coherent state formulation discussed above we used Euclidean time and

$$\exp \{-S_E\} = \exp \left\{ - \int (\bar{\eta} \partial_\tau \eta + H(\bar{\eta}, \eta)) d\tau \right\}$$

and passing to Minkowski time $t = -i\tau$ we find $e^{i \int (\bar{\eta} i \partial_t \eta - H(\bar{\eta}, \eta)) dt}$

For Dirac fermions in Minkowski space-time we have $\mathcal{L} = \bar{\psi}(i\partial - m)\psi = \psi^\dagger i \partial_t \psi + \dots$

(pay attention that $\bar{\psi} = -\psi^\dagger \gamma^0$ in Zinn-Justin book, instead of the usual $\bar{\psi} = \psi^\dagger \gamma^0$) hence

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \psi^\dagger \quad \text{and} \quad \mathcal{H}(\psi^\dagger, \psi) = p \dot{q} - \mathcal{L} = i \psi^\dagger \partial_t \psi - \psi^\dagger (\gamma^0 i \partial - m \gamma^0) \psi = \psi^\dagger (-i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} + \gamma^0 m) \psi$$

where $\partial = \gamma^0 \partial_0 + \vec{\gamma} \cdot \vec{\nabla}$ and we denote by $\vec{\gamma}$ the vector γ^i , $i=1,2,3$.

Since for fermions $\{\psi, \psi\} = i$, we have $\{\psi^\dagger, \psi\} = 1$, i.e. the same commutation relation of $a, a^\dagger$

and we can thus build the coherent states associated to ψ . In the coherent-state path-integral we thus have (note: $\bar{\eta}$ is the element of the Grassmann algebra, NOT $\eta^\dagger \gamma^0$)

$$i\bar{\eta} \partial_t \eta - H(\bar{\eta}, \eta) = \bar{\eta} \gamma^0 \gamma^0 i \partial_t \eta + \bar{\eta} (i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} - \gamma^0 m) \eta = \bar{\eta} \gamma^0 (i \not{\partial} - m) \eta$$

to be used to compute correlators of $\psi^\dagger, \psi$:

$$Z(\Theta, \bar{\Theta}) = \int D(\bar{\eta}, \eta) \exp \left\{ i \int (\bar{\eta} \gamma^0 (i \not{\partial} - m) \eta + \bar{\Theta} \eta + \bar{\eta} \Theta) dt d^d x \right\}$$

$$\langle 0 | T(\psi_i^\dagger(x) \psi_j(y)) | 0 \rangle = \frac{\int D(\bar{\eta}, \eta) \bar{\eta}_i(x) \eta_j(y) e^{i \int \bar{\eta} \gamma^0 (i \not{\partial} - m) \eta d^d x}}{\int D(\bar{\eta}, \eta) e^{i \int \bar{\eta} \gamma^0 (i \not{\partial} - m) \eta d^d x}} = \frac{1}{Z} \frac{\delta}{\delta \bar{\Theta}_j(y)} \frac{\delta}{\delta \Theta_i(x)} Z(\Theta, \bar{\Theta}) \Big|_{\Theta = \bar{\Theta} = 0}$$

If we are interested in computing expectation values of $\bar{\psi} = \psi^\dagger \gamma^0$ and ψ we can perform the change of variable $\bar{\eta} \rightarrow \bar{\eta} \gamma^0$ in the path-integral:

$$\langle 0 | T(\bar{\psi}(x) \psi(y)) | 0 \rangle = \frac{\int D(\bar{\eta}, \eta) \bar{\eta}(x) \eta(y) e^{i \int \bar{\eta} (i \not{\partial} - m) \eta d^d x}}{\int D(\bar{\eta}, \eta) e^{i \int \bar{\eta} (i \not{\partial} - m) \eta d^d x}}$$

and use the generating functional

$$Z(\Theta, \bar{\Theta}) = \int D(\bar{\eta}, \eta) \exp \left\{ i \int (\bar{\eta} (i \not{\partial} - m) \eta + \bar{\Theta} \eta + \bar{\eta} \Theta) dt d^d x \right\}$$

which, apart from the fact that integration variables are Grassmann numbers, is what could have been guessed, since the exponent is just $i \int \mathcal{L} d^d x$.

We have seen before that

$$\int \prod_i d\bar{\psi}_i d\psi_i \exp \left\{ - \sum_{ij} \bar{\psi}_i A_{ij} \psi_j + \sum_i \bar{\Theta}_i \psi_i + \sum_i \bar{\psi}_i \Theta_i \right\} = (\det A) \exp \left\{ \sum_{ij} \bar{\Theta}_i (A^{-1})_{ij} \Theta_j \right\}$$

thus

$$Z(\Theta, \bar{\Theta}) = Z[0] \exp \left\{ \int i \bar{\Theta}(x) \left[\frac{1}{-i(i \not{\partial} - m)} \right]_{xy} i \bar{\Theta}(y) d^d x d^d y \right\} = Z[0] \exp \left\{ -i \int \bar{\Theta}(x) \left(\frac{1}{i \not{\partial} - m} \right)_{xy} \Theta(y) d^d x d^d y \right\}$$

For the two point function we have

$$\begin{aligned} \langle 0 | T(\psi(x) \bar{\psi}(y)) | 0 \rangle &= \frac{(-i)^2}{Z[0]} \frac{\delta^2}{\delta \Theta(y) \delta \bar{\Theta}(x)} Z \Big|_{\Theta = \bar{\Theta} = 0} = \left(\frac{i}{i \not{\partial} - m} \right)_{xy} = i \int \frac{d^d k}{(2\pi)^d} \frac{1}{k - m} e^{-ik(x-y)} = \\ &= i \int \frac{d^d k}{(2\pi)^d} \frac{k + m}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} = i S_F(x-y) \end{aligned}$$

built in such a way that $\int (i \not{\partial} - m)_{xx} (i \not{\partial} - m)_{xx} d^d x = \delta(x-y)$
 as for scalars it is needed to avoid the pole when passing from Euclidean to Minkowski

For the $2n$ -points correlation function we have

$$\langle 0 | T \psi_{i_1} \bar{\psi}_{j_1} \dots \psi_{i_n} \bar{\psi}_{j_n} | 0 \rangle = \frac{(-i)^{2n}}{Z[0]} \frac{\delta}{\delta \Theta_{j_1}} \frac{\delta}{\delta \bar{\Theta}_{i_1}} \dots \frac{\delta}{\delta \Theta_{j_n}} \frac{\delta}{\delta \bar{\Theta}_{i_n}} Z(\Theta, \bar{\Theta}) \Big|_{\Theta = \bar{\Theta} = 0} =$$

$$\begin{aligned}
&= (-i)^{2m} \int d\theta_{j_1} d\bar{\theta}_{i_1} \dots d\theta_{j_m} d\bar{\theta}_{i_m} \exp \left\{ - \int \bar{\theta}(x) i S_F(x-y) \theta(y) d^D x d^D y \right\} \Big|_{\theta=\bar{\theta}=0} = \\
&= (-i)^{2m} \int d\bar{\theta}_{i_1} d\theta_{j_1} \dots d\bar{\theta}_{i_m} d\theta_{j_m} \exp \left\{ \int \bar{\theta}(x) i S_F(x-y) \theta(y) d^D x d^D y \right\} \Big|_{\theta=\bar{\theta}=0} = \\
&= (-i)^{2m} \det \left[-i S_F(x-y) \right]_{i_k j_l} = (-i)^{2m} \prod_{\substack{\text{perm} \\ j_1 \dots j_m}} (-1)^{\sigma} (-i S_F(x_{i_1} - y_{j_{P_1}})) \dots (-i S_F(x_{i_m} - y_{j_{P_m}})) = \\
&= \prod_{\substack{\text{perm} \\ j_1 \dots j_m}} (-1)^{\sigma} i S_F(x_{i_1} - y_{j_{P_1}}) \dots i S_F(x_{i_m} - y_{j_{P_m}}) = \prod_{\substack{\text{perm} \\ j_1 \dots j_m}} (-1)^{\sigma} \langle 0 | T \psi_{i_1} \bar{\psi}_{j_{P_1}} | 0 \rangle \dots \langle 0 | T \psi_{i_m} \bar{\psi}_{j_{P_m}} | 0 \rangle
\end{aligned}$$

which is just Wick theorem.

Euclidean γ matrices: $\gamma_E^0 = \gamma^0$, $\gamma_E^j = -i \gamma^j$, hence γ_E^μ are Hermitian and the Euclidean form of the Clifford algebra is $\{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu}$. Using γ_E the Hinkowski action is written

$$\begin{aligned}
S &= \int \bar{\psi}(i \not{\partial} - m) \psi dt d^D x = \int \bar{\psi}(i \gamma^0 \partial_0 + i \gamma^j \partial_j - m) \psi dt d^D x = \int \bar{\psi}(i \gamma_E^0 \partial_0 - \vec{\gamma}_E \cdot \vec{\nabla} - m) \psi dt d^D x \\
iS &= i \int \bar{\psi} i \gamma_E^0 \partial_t \psi dt d^D x - i \int \bar{\psi} (\vec{\gamma}_E \cdot \vec{\nabla} + m) \psi dt d^D x \stackrel{t=-i\tau}{=} \\
&= - \int \bar{\psi} (\gamma_E^0 \partial_\tau + \vec{\gamma}_E \cdot \vec{\nabla} + m) \psi d\tau d^D x = - \int \bar{\psi} (\not{\partial}_E + m) \psi d\tau d^D x = -S_E \\
\text{hence } S_E &= \int \bar{\psi} (\not{\partial}_E + m) \psi d^D x_E
\end{aligned}$$

The Euclidean correlation generating functional is ($k = \not{\partial}_E + m$)

$$Z(\theta, \bar{\theta}) = \int \mathcal{D}(\bar{\psi}, \psi) \exp \left\{ - \int (\bar{\psi} k \psi - \bar{\theta} \eta - \bar{\eta} \theta) \right\} = Z(0) e^{\int \bar{\theta}(x) (k^{-1})_{xy} \theta(y) d^D x d^D y}$$

the Euclidean two point function is

$$\begin{aligned}
\langle \psi(x) \bar{\psi}(y) \rangle &= \frac{1}{Z(0)} \frac{\delta}{\delta \bar{\theta}(y)} \frac{\delta}{\delta \bar{\theta}(x)} Z \Big|_{\theta=\bar{\theta}=0} = (k^{-1})_{xy} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{-i \not{p} + m} e^{-ip(x-y)} = \\
&= \int \frac{d^D p}{(2\pi)^D} \frac{i \not{p} + m}{p^2 + m^2} e^{-ip(x-y)} \quad \text{and} \quad \langle \bar{\psi}(y) \psi(x) \rangle = -\langle \psi(x) \bar{\psi}(y) \rangle
\end{aligned}$$

and Wick theorem has exactly the same form as in Minkowsky spacetime.

If the action is of the form $S_E = \int \bar{\psi} k \psi + \int V(\bar{\psi}, \psi)$ we can proceed as in the bosonic

$$\text{case and write } Z(\theta, \bar{\theta}) = \exp \left\{ - \int d^D x V \left(-\frac{\delta}{\delta \bar{\theta}}, \frac{\delta}{\delta \theta} \right) \right\} \int \mathcal{D}(\bar{\psi}, \psi) e^{- \int (\bar{\psi} k \psi + \bar{\eta} \theta + \bar{\theta} \eta)}$$

so everything reduces to the computation of Gaussian functional integrals.

The generator of connected correlators is again $W(\theta, \bar{\theta}) = \log Z(\theta, \bar{\theta})$ and the effective action

is defined by the Legendre transformation

$$\Gamma(\psi, \bar{\psi}) + W(\theta, \bar{\theta}) = \int (\bar{\theta} \psi + \bar{\psi} \theta) d^D x$$

$$\text{with } \psi(x) = \frac{\delta W}{\delta \bar{\theta}(x)} \quad \text{and} \quad \bar{\psi}(x) = - \frac{\delta W}{\delta \theta(x)}, \quad \text{from which } \frac{\delta \Gamma}{\delta \bar{\psi}(x)} = \theta(x) \quad \text{and} \quad \frac{\delta \Gamma}{\delta \psi(x)} = - \bar{\theta}(x) \text{ follow.}$$

In the four dimensional spacetime we have $\{\gamma_5, \gamma^\mu\} = 0$ hence if $i \not{\partial} \psi = \lambda \psi$

then $i \not{\partial} \gamma_5 \psi = -i \gamma_5 \not{\partial} \psi = -\lambda \gamma_5 \psi$, hence if λ is an eigenvalue of $i \not{\partial}$ also $-\lambda$ is. Thus

$$\det(i\not{p}-m) = \prod_m (\lambda_m - m)(-\lambda_m - m) = \prod_m (-\lambda_m^2 - m^2) = \det(-i\not{p}-m)$$

In the Euclidean case we have analogously

$$\det(\not{p}_E + m) = \prod_m (-\lambda_m^2 + m^2) = \det(-\not{p}_E + m)$$

but now γ_E matrices are Hermitian, hence $\not{p}_E \psi = \lambda \psi$ implies $\lambda = i\mu$ hence $\det(\not{p}_E + m) > 0$, which also implies

$$\det(\not{p}_E + m) = [\det(\not{p}_E + m) \det(-\not{p}_E + m)]^{1/2} = [\det(-\not{p}_E^2 + m^2) \cdot \mathbb{1}_{4 \times 4}]^{1/2} = [\det(-\not{p}_E^2 + m^2)]^2$$

↑
Klein-Gordon determinant

Note also that $\det(\not{p}_E + m) = \det(-\not{p}_E + m) = \det(-\not{p}_E - m)$, hence the sign of the fermionic term is irrelevant.

ASYMPTOTIC FIELDS

For a free scalar field we have $\phi_{\text{free}}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} (a(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx})$, where $p^2 = m^2$ and creation and annihilation operators are

normalized in such a way that $[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \delta(\vec{p} - \vec{q})$. If we introduce the notation $\overleftrightarrow{\partial}_0 = \overrightarrow{\partial}_0 - \overleftarrow{\partial}_0$ it is simple to see that

$$a(\vec{p}) = i \int \frac{e^{ipx}}{\sqrt{2E_p}} \overleftrightarrow{\partial}_0 \phi_{\text{free}}(x) d^3x \quad (\text{integral at fixed timeslice})$$

Since ϕ_{free} satisfies $(\square + m^2)\phi_{\text{free}} = 0$ it is simple to see that $a(\vec{p})$ is time independent

$$\begin{aligned} \partial_0 a(\vec{p}) &= \frac{i}{\sqrt{2E_p}} \partial_0 \int (e^{ipx} \partial_0 \phi_{\text{free}}(x) - \phi_{\text{free}}(x) \partial_0 e^{ipx}) d^3x = \partial_0^2 \phi_{\text{free}} = (\nabla^2 - m^2) \phi_{\text{free}} \\ &= \frac{i}{\sqrt{2E_p}} \int (e^{ipx} \partial_0^2 \phi_{\text{free}}(x) - \phi_{\text{free}}(x) \partial_0^2 e^{ipx}) d^3x \quad \text{and integrating by parts} \\ &= \frac{i}{\sqrt{2E_p}} \int \phi_{\text{free}}(x) (\nabla^2 - m^2 - \partial_0^2) e^{ipx} d^3x = \frac{i}{\sqrt{2E_p}} \int \phi_{\text{free}}(x) (E_p^2 - \vec{p}^2 - m^2) e^{ipx} d^3x = 0 \end{aligned}$$

If we use the relativistic normalization of states $|\vec{p}\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle$ we then have immediately $\langle 0 | \phi_{\text{free}}(x) | \vec{p} \rangle = e^{-ipx}$

Let us now consider an interacting field: from $\phi(x) = e^{i\vec{p}x} \phi(0) e^{-i\vec{p}x}$ we have

$\langle 0 | \phi(x) | \vec{p} \rangle = \langle 0 | \phi(0) | \vec{p} \rangle e^{-ipx}$, hence in particular we have $(\square + m^2) \langle 0 | \phi(x) | \vec{p} \rangle = 0$, from which the fact that $i \int \frac{e^{ipx}}{\sqrt{2E_p}} \overleftrightarrow{\partial}_0 \langle 0 | \phi(x) | \vec{p} \rangle$ is independent of time follows.

We now assume that

$$\begin{aligned} \langle 0 | \phi(x) | \vec{p} \rangle &\xrightarrow{x^0 \rightarrow \pm\infty} \mathcal{Z}_{\text{out/in}}^{1/2} \langle 0 | \phi_{\text{out/in}} | \vec{p} \rangle \quad \text{where } \phi_{\text{out}} \text{ and } \phi_{\text{in}} \text{ are asymptotic free fields. Since} \\ i \int \frac{e^{ipx}}{\sqrt{2E_p}} \overleftrightarrow{\partial}_0 \langle 0 | \phi(x) | \vec{p} \rangle &\text{ is time independent (just as } i \int \frac{e^{ipx}}{\sqrt{2E_p}} \overleftrightarrow{\partial}_0 \langle 0 | \phi_{\text{out/in}}(x) | \vec{p} \rangle) \text{ we thus have} \\ i \int \frac{e^{ipx}}{\sqrt{2E_p}} \overleftrightarrow{\partial}_0 \langle 0 | \phi(x) | \vec{p} \rangle &= \mathcal{Z}_{\text{out/in}}^{1/2} i \int \frac{e^{ipx}}{\sqrt{2E_p}} \overleftrightarrow{\partial}_0 \langle 0 | \phi_{\text{out/in}}(x) | \vec{p} \rangle \end{aligned}$$

and if the asymptotic fields are normalized as free fields, $\langle 0 | \phi_{\text{out}}(x) | \vec{p} \rangle = \langle 0 | \phi_{\text{in}}(x) | \vec{p} \rangle = e^{-ipx}$, we get $\mathcal{Z}_{\text{in}} = \mathcal{Z}_{\text{out}} \equiv \mathcal{Z}$ and $\langle 0 | \phi(x) | \vec{p} \rangle = \mathcal{Z}^{1/2} e^{-ipx}$. In general $\mathcal{Z}^{1/2}$ connects all matrix elements of $\phi(x)$ for $x^0 \rightarrow \pm\infty$ and matrix elements of $\phi_{\text{out/in}}$.

KÄLLÉN-LEHMAN SPECTRAL REPRESENTATION

We want to study the T-ordered two point function for interacting fields $\langle 0 | T \phi(x) \phi(x') | 0 \rangle$, with

$$T \phi(x) \phi(x') = \theta(x_0 - x'_0) \phi(x) \phi(x') + \theta(x'_0 - x_0) \phi(x') \phi(x)$$

Let us start from $\langle 0 | \phi(x) \phi(x') | 0 \rangle = \sum_n \langle 0 | \phi(x) | n \rangle \langle n | \phi(x') | 0 \rangle = \sum_n e^{-i p_n (x - x')} |\langle 0 | \phi(0) | n \rangle|^2$
 $\phi(x) = e^{i\vec{p}x} \phi(0) e^{-i\vec{p}x}$

If we assume for the sake of the simplicity that $\langle 0 | \phi(0) | 0 \rangle = 0$ we can remove the vacuum ($n=0$) from the sum (which is equivalent to considering the connected two point function), hence

$$\langle 0 | \phi(x) \phi(x') | 0 \rangle = \int d^4 q e^{-i q (x-x')} \sum_{n>0} |\langle 0 | \phi(0) | n \rangle|^2 \delta^4(q - P_n) \equiv \int \frac{d^4 q}{(2\pi)^3} e^{-i q (x-x')} \rho(q)$$

↑
momentum of the state $|n\rangle$

where we introduced the spectral density $\rho(q) = (2\pi)^3 \sum_{n>0} |\langle 0 | \phi(0) | n \rangle|^2 \delta^4(q - P_n)$.

The spectral density $\rho(q)$ is Lorentz invariant, hence it can be function only of q^2 , moreover it is nonzero only in the forward light cone ($q^2 > 0, q_0 \geq 0$), since it is related to physical momenta, hence $\rho(q) = \rho(q^2) \Theta(q_0)$.

We thus have (we use $x'=0$ for the sake of the simplicity)

$$\begin{aligned} \langle 0 | T \phi(x) \phi(0) | 0 \rangle &= \int \frac{d^4 q}{(2\pi)^3} \rho(q^2) \Theta(q_0) [\Theta(x_0) e^{-i q x} + \Theta(-x_0) e^{i q x}] \quad \leftarrow \text{inserting } 1 = \int d\sigma \delta(q^2 - \sigma) \\ &= \int_0^\infty d\sigma \rho(\sigma) \int \frac{d^4 q}{(2\pi)^3} \delta(q^2 - \sigma) \Theta(q_0) [\Theta(x_0) e^{-i q x} + \Theta(-x_0) e^{i q x}] \quad \times \end{aligned}$$

To understand what is the expression multiplying $\rho(\sigma)$ under the integral let us consider single particle states. For single particle states, using $\langle 0 | \phi(x) | \vec{p} \rangle = 2^{1/2} e^{-i p x}$ and

$$\begin{aligned} \sum_{\vec{p}} |n\rangle \langle n| &= \int \frac{d^3 p}{(2\pi)^3 2E_p} |\vec{p}\rangle \langle \vec{p}| \quad (|\vec{p}\rangle \text{ with relativistic normalization!}) \quad \text{we have} \\ \rho(q^2) \big|_{\vec{p}} \Theta(q_0) &= (2\pi)^3 \Theta(q_0) \int \frac{d^3 p}{(2\pi)^3 2E_p} |\langle 0 | \phi(0) | \vec{p} \rangle|^2 \delta^4(q - p) = \Theta(q_0) 2 \int \frac{d^3 p}{2E_p} \delta^4(q - p) = \\ &= 2 \Theta(q_0) \frac{1}{2E_p} \delta(q_0 - p_0) = 2 \Theta(q_0) \delta(q^2 - m^2) \end{aligned}$$

where in the last step we used

$$\delta(q^2 - m^2) \Theta(q_0) = \Theta(q_0) \frac{1}{2E_q} [\delta(q_0 - E_q) + \delta(q_0 + E_q)] = \Theta(q_0) \frac{1}{2E_q} \delta(q_0 - E_q).$$

Hence, by separately considering single particle states and multiparticle states we have

$$\rho(q) = \rho(q^2) \Theta(q_0) = 2 \delta(q^2 - m^2) \Theta(q_0) + (2\pi)^3 \sum_{\text{multi-p.}} |\langle 0 | \phi(0) | n \rangle|^2 \delta^4(q - P_n). \quad \times^2$$

Let us now go back to eq $\times$: everything multiplying $\rho(\sigma)$ is independent of the dynamics, only come from kinematical factors. In particular it can be computed in free field theory. In that case we have

$$\langle 0 | T \phi(x) \phi(0) | 0 \rangle = i \Delta(x) = i \int \frac{d^4 q}{(2\pi)^4} \frac{e^{-i q x}}{q^2 - m^2 + i\epsilon} \quad \text{and from } \times^2 \text{ we see that}$$

for a free field $\rho(q^2) = \delta(q^2 - m^2)$, hence

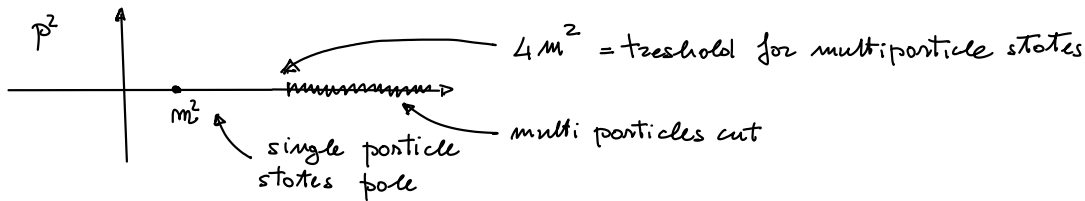
we can write the free propagator as

$$\langle 0 | T \phi(x) \phi(0) | 0 \rangle = i \Delta(x) = \int_0^\infty d\sigma \delta(\sigma - m^2) \underbrace{\int \frac{d^4 q}{(2\pi)^4} \frac{i e^{-i q x}}{q^2 - \sigma + i\epsilon}}_{\equiv i \Delta(x, \sigma)}$$

and identify everything which multiplies $\rho(\sigma)$ with $\nearrow$ also in the interacting case.

We thus finally get $\langle 0 | T \phi(x) \phi(0) | 0 \rangle = \int d\sigma \rho(\sigma) i \Delta(x, \sigma)$

and the analytical structure of the two point function in the complex p^2 plane is



and close to a single particle pole $\langle 0 | T \phi(x) \phi(0) | 0 \rangle \simeq \mathcal{Z} i \Delta(x, m^2)$.

Note that we introduced the spectral function to study the T-ordered two point function, but the same spectral function enters all two point functions, and in particular the causal two point function that is needed in linear response theory to define kinetic coefficients (conductibility, viscosity, ...) by Kubo formulae.

Using the spectral representation to write $\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$, taking a time derivative and using canonical commutation relations it is possible to get the sum rule

$$1 = \mathcal{Z} + \int_{m^2}^{\infty} \rho(\sigma) d\sigma$$

from which $0 \leq \mathcal{Z} \leq 1$, see ITZYKSON, ZUBER "Quantum field theory" § 5.1.2.

LSZ REDUCTION FORMULAE (LEHMANN, SYMANZYK, ZIMMERMANN)

The LSZ reduction formulae connect S matrix elements

$$\langle \vec{p}_1, \dots, \vec{p}_n, \text{in} | S | \vec{q}_1, \dots, \vec{q}_m, \text{in} \rangle = \langle \vec{p}_1, \dots, \vec{p}_n, \text{out} | \vec{q}_1, \dots, \vec{q}_m, \text{in} \rangle$$

with T-ordered vacuum expectation values, and in the following we consider just $n=m=2$.

We previously saw that

$$a(\vec{p}) = i \int \frac{e^{ipx}}{\sqrt{2E_p}} \overleftrightarrow{\partial}_0 \phi_{\text{free}}(x) d^3x \quad (\text{independent of } x^0)$$

Using the relativistic normalization of the states $|\vec{p}, \text{in}\rangle = \sqrt{2E_p} a_{\text{in}}^+(\vec{p}) | 0 \rangle$, the expression

$$a_{\text{in}}^+(\vec{q}_1) = -i \int \frac{d^3x_1}{\sqrt{2E_{q_1}}} e^{-iq_1x_1} \overleftrightarrow{\partial}_0 \phi_{\text{in}}(x_1)$$

and the definition of $\mathcal{Z}$ we have

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_2, \text{out} | \vec{q}_1, \vec{q}_2, \text{in} \rangle &= -i \int d^3x_1 e^{-iq_1x_1} \overleftrightarrow{\partial}_{x_1^0} \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi_{\text{in}}(x_1) | q_2, \text{in} \rangle \\ &= -i \lim_{x_1^0 \rightarrow -\infty} \frac{1}{\mathcal{Z}^{1/2}} \int d^3x_1 e^{-iq_1x_1} \overleftrightarrow{\partial}_{x_1^0} \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi(x_1) | q_2, \text{in} \rangle \end{aligned}$$

independent of x_1^0
hence we consider $x_1^0 \rightarrow -\infty$

$$\text{In general we have } \lim_{t_f \rightarrow +\infty} \lim_{t_i \rightarrow -\infty} \int_{t_i}^{t_f} dt \frac{\partial}{\partial t} \int d^3x f(x, t) = \left(\lim_{t_f \rightarrow +\infty} - \lim_{t_i \rightarrow -\infty} \right) \int d^3x f(x, t) \quad \times$$

hence we can use

$$\lim_{t_i \rightarrow -\infty} \int d^3x f(x, t_i) = \lim_{t_f \rightarrow \infty} \int d^3x f(x, t_f) - \int_{-\infty}^{+\infty} dt \int d^3x \frac{\partial}{\partial t} f(x, t)$$

to write

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_2, \text{out} | \vec{q}_1, \vec{q}_2, \text{in} \rangle &= -i \lim_{x_2^0 \rightarrow +\infty} \frac{1}{2^{1/2}} \int d^3x_1 e^{-i q_1 x_1} \overleftrightarrow{\partial}_{x_1^0} \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi(x_1) | \vec{q}_2, \text{in} \rangle + \\ &\quad + \frac{i}{2^{1/2}} \int d^4x_1 \partial_{x_1^0} \left[e^{-i q_1 x_1} \overleftrightarrow{\partial}_{x_1^0} \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi(x_1) | \vec{q}_2, \text{in} \rangle \right] = \\ &= \underbrace{\langle \vec{p}_1, \vec{p}_2, \text{out} | a_{\text{out}}^+(\vec{q}_1) | \vec{q}_2, \text{in} \rangle}_{\text{disc. terms}} + \frac{i}{2^{1/2}} \int d^4x_1 \partial_{x_1^0} \left[e^{-i q_1 x_1} \overleftrightarrow{\partial}_{x_1^0} \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi(x_1) | \vec{q}_2, \text{in} \rangle \right] \end{aligned}$$

$$= (2\pi)^3 \delta E_{\vec{p}_1} \delta^3(\vec{p}_1 - \vec{q}_1) \langle \vec{p}_2, \text{out} | \vec{q}_1, \text{in} \rangle + (2\pi)^3 \delta E_{\vec{p}_2} \delta^3(\vec{p}_2 - \vec{q}_2) \langle \vec{p}_1, \text{out} | \vec{q}_2, \text{in} \rangle$$

terms which do not contribute to the nontrivial part of the S matrix, usually denoted by T :

$$S = 1 + iT, \quad \langle \text{out} | T | \text{in} \rangle = (2\pi)^4 \delta^4(P_{\text{out}} - P_{\text{in}}) \langle \text{out} | M | \text{in} \rangle \quad \text{where } M \text{ acts on energy shell}$$

(see e.g. ITZYKSON, ZUBER "Quantum field theory" § 5.1.1, WEINBERG "The quantum theory of fields vol 1" § 3.2-3.3)

Since $q_1^2 = m^2$ we have $(\square_{x_1} + m^2) e^{-i q_1 x_1} = 0$ hence

$$\begin{aligned} \int d^4x_1 \partial_{x_1^0} \left[e^{-i q_1 x_1} \overleftrightarrow{\partial}_{x_1^0} \langle \beta, \text{out} | \phi(x_1) | \alpha, \text{in} \rangle \right] &= (-\nabla^2 + m^2) e^{-i q_1 x_1} \langle \beta, \text{out} | \phi(x_1) | \alpha, \text{in} \rangle \\ &= \int d^4x_1 \left[e^{-i q_1 x_1} \partial_{x_1^0}^2 \langle \beta, \text{out} | \phi(x_1) | \alpha, \text{in} \rangle + \underbrace{\left(-\partial_{x_1^0}^2 e^{-i q_1 x_1} \right)}_{\text{integrating by parts}} \langle \beta, \text{out} | \phi(x_1) | \alpha, \text{in} \rangle \right] \\ &= \int d^4x_1 e^{-i q_1 x_1} (\square_{x_1} + m^2) \langle \beta, \text{out} | \phi(x_1) | \alpha, \text{in} \rangle \end{aligned}$$

and we arrived to

$$\langle \vec{p}_1, \vec{p}_2, \text{out} | \vec{q}_1, \vec{q}_2, \text{in} \rangle = \frac{i}{2^{1/2}} \int d^4x e^{-i q_1 x} (\square_{x_1} + m^2) \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi(x_1) | \vec{q}_2, \text{in} \rangle + \text{disc. terms}$$

We can now proceed analogously with $\vec{p}_2$:

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi(x_1) | \vec{q}_2, \text{in} \rangle &= \langle \vec{p}_2, \text{out} | a_{\text{out}}(\vec{p}_1) \phi(x_1) | \vec{q}_2, \text{in} \rangle = \\ &= \lim_{y_1^0 \rightarrow +\infty} \frac{i}{2^{1/2}} \int d^3y_1 e^{i p_1 y_1} \overleftrightarrow{\partial}_{y_1^0} \langle \vec{p}_2, \text{out} | \phi(y_1) \phi(x_1) | \vec{q}_2, \text{in} \rangle \end{aligned}$$

Using the general relation * with $f = \langle \vec{p}_2, \text{out} | T \phi(y_1) \phi(x_1) | \vec{q}_2, \text{in} \rangle$ we get

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_2, \text{out} | \phi(x_1) | \vec{q}_2, \text{in} \rangle &= \langle \vec{p}_2, \text{out} | \phi(x_1) a_{\text{in}}(\vec{p}_1) | \vec{q}_2, \text{in} \rangle + \\ &\quad + \frac{i}{2^{1/2}} \int d^4y_1 e^{i p_1 y_1} (\square_{y_1} + m^2) \langle \vec{p}_2, \text{out} | T \phi(y_1) \phi(x_1) | \vec{q}_2, \text{in} \rangle \end{aligned}$$

and the first term generates further disconnected terms, hence

$$\langle \vec{p}_1, \vec{p}_2, \text{out} | \vec{q}_1, \vec{q}_2, \text{in} \rangle = \left(\frac{i}{2^{1/2}} \right)^2 \int d^4x_1 d^4y_1 e^{i p_1 y_1 - i q_1 x_1} (\square_{y_1} + m^2) (\square_{x_1} + m^2) \langle \vec{p}_2, \text{out} | T \phi(y_1) \phi(x_1) | \vec{q}_2, \text{in} \rangle + \text{disc.}$$

and analogously one gets

$$\begin{aligned} \langle \vec{p}_1, \dots, \vec{p}_m, \text{out} | \vec{q}_1, \dots, \vec{q}_m, \text{in} \rangle &= \left(\frac{i}{2^{1/2}} \right)^{m+m} \int d^4y_1 \dots d^4y_m d^4x_1 \dots d^4x_m e^{i \sum_{k=1}^m p_k y_k - i \sum_{k=1}^m q_k x_k} \\ &\quad \cdot (\square_{y_1} + m^2) \dots (\square_{y_m} + m^2) (\square_{x_1} + m^2) \dots (\square_{x_m} + m^2) \langle 0 | T \phi(y_1) \dots \phi(y_m) \phi(x_1) \dots \phi(x_m) | 0 \rangle + \text{disc} \end{aligned}$$

where disconnected terms take care of the cases in which at least one particle does not interact.

These are the reduction formulae.

In Fourier transform $i(\Box_{y_k} + m^2) \rightarrow i(-p_k^2 + m^2)$ and $i(\Box_{x_k} + m^2) \rightarrow i(-q^2 + m^2)$

hence S matrix elements are the residues at the on-shell multiple pole of the T -ordered correlation functions of the field normalized as a free field (note the $\frac{1}{2^{1/2}}$ factors).

Some comments

- 1) If we just want the totally connected term of the S -matrix, characterized by $\delta^4(\sum p_i - \sum q_i)$, we can consider the connected correlation function in the LSZ reduction formulae (see ITZYKSON, ZUBER "Quantum field theory" § 5.1.5). For the $2 \rightarrow 2$ case this is not necessary since two point functions have a single pole on-shell, and not a double pole, hence they do not contribute.
- 2) The field ϕ is not required to be the fundamental field associated to asymptotic excitations, it is only important that it couples to them: $\langle 0 | \phi(x) | \vec{p} \rangle \neq 0$ (this is important for effective field theories)
- 3) For fermions see ITZYKSON, ZUBER "Quantum field theory" § 5.1.6
- 4) From LSZ reduction formulae and Kallen-Lehmann representation it follows that radiative corrections to external legs only contribute to S -matrix elements through Z .

QED

GENERAL FACTS

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{D} - m) \psi = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{\partial} - m) \psi - e \bar{\psi} \not{A} \psi$$

$$\text{where } D_\mu = \partial_\mu + ie A_\mu \text{ and } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Lagrangian is invariant under the global transformation $\begin{cases} \psi(x) \rightarrow e^{i\theta} \psi(x) \\ \bar{\psi}(x) \rightarrow e^{-i\theta} \bar{\psi}(x) \end{cases}$ whose infinitesimal form is $\psi(x) \rightarrow \psi(x) + i\theta \psi(x)$.

If the Lagrangian density is invariant under the infinitesimal transformation $\varphi^a \rightarrow \varphi^a + i\theta f^a$ then the current $J^\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} f^a$ is conserved (see e.g. WEINBERG "The quantum theory of fields 1" § 4.3). In QED $J^\mu = \bar{\psi} \gamma^\mu \psi$.

The classical equations of motion $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} - \frac{\partial \mathcal{L}}{\partial \varphi^a} = 0$ give $\begin{cases} \partial_\mu F^{\mu\nu} = e J^\nu \\ (i \not{D} - m) \psi = 0 \end{cases}$

Gauge invariance: $\psi(x) \rightarrow \psi(x) e^{i\theta(x)}$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{-i\theta(x)}$, $A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \theta$, in such a way that $D_\mu \psi(x) \rightarrow (\partial_\mu + ie A_\mu - i \partial_\mu \theta) [e^{i\theta(x)} \psi(x)] = e^{i\theta(x)} [\partial_\mu + ie A_\mu] \psi(x) = e^{i\theta(x)} D_\mu \psi(x)$

To pass to Euclidean space we use $x^0 = -ix^4$, $\gamma_E^0 = \gamma^0$, $\gamma_E^j = -i \gamma^j$ and $A_0 = i A_E^4$, $A_j = A_E^j$

(note that to maintain gauge invariance A_0 has to transform as ∂_0 and A_j as ∂_j):

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = i(\partial_4 A_E^i - \partial_i A_E^4) \quad (\text{in Euclidean space covariant = contragradient})$$

$$\text{hence } -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} F_{0i} F_{0i} + \frac{1}{4} F_{i0} F_{i0} - \frac{1}{4} F_{ij} F_{ij} \text{ gives}$$

$$i S_H = i \int d^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \rightarrow i(-i) \int d^4 x_E \left(-\frac{1}{4} F_{4i} F_{4i} - \frac{1}{4} F_{i4} F_{i4} - \frac{1}{4} F_{ij} F_{ij} \right) = -\frac{1}{4} \int d^4 x_E F_{\mu\nu} F_{\mu\nu} = -S_E$$

hence $S_E = \frac{1}{4} \int F_{\mu\nu} F_{\mu\nu} d^4 x_E$ for gauge fields and we have seen before that the Euclidean Lagrangian density is $\bar{\psi} (\not{D} + m) \psi$ for the free fermionic field. Let us finally consider the interaction term:

$$i S_H = i \int d^4 x \left(-e \bar{\psi} A_\mu \gamma^\mu \psi \right) = i(-i) \int d^4 x_E \left(-i e \bar{\psi} A_E^\mu \gamma_E^\mu \psi \right) = - \int d^4 x_E i e \bar{\psi} \not{A}_E \psi = -S_E$$

hence finally

$$\mathcal{L}_E = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} (\not{D} + m) \psi + ie \bar{\psi} \not{A} \psi = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} (\not{D} + m) \psi \quad \text{with } D_\mu = \partial_\mu + ie A_\mu$$

Problems of canonical quantization: $\pi_0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} = 0$, hence it is impossible to impose the equal time canonical commutation relations $[\pi_0(t, \vec{x}), A_0(t, \vec{y})] = -i \hbar \delta(\vec{x} - \vec{y})$. Possible solutions

1) Abandon manifest Lorentz invariance, using Coulomb or temporal gauge, see e.g.

ZINN-SUSTIN "Quantum field theory and critical phenomena" § 18.4, WEINBERG "The quantum theory of fields 1" § 8.3, FRADKIN "Quantum field theory an integrated approach" § 9.3

2) Keep manifest Lorentz invariance and abandon norm positivity, see e.g. ITZYKSON-ZUBER "Quantum Field Theory" § 3.2

GAUGE FIXING

The problem of canonical quantization is strictly related to gauge invariance, i.e. to the use of a redundant representation. This creates problems also in the path-integral approach, since we get formally divergent integrals, whose divergence is due to the presence of "flat" directions in the fields space, a fact which causes troubles also for the computation of the photon propagator.

$$\frac{1}{4} \int F_{\mu\nu} F_{\mu\nu} d^4x = \frac{1}{4} \int (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) d^4x = \frac{1}{2} \int (\partial_\mu A_\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu) d^4x =$$

$$= \frac{1}{2} \int A_\nu(x) (\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial_\alpha^2) A_\mu(x) d^4x = \frac{1}{2} \int A_\mu(x) K_{\mu\nu}(x, y) A_\nu(y) d^4x d^4y$$

$$K_{\mu\nu}(x, y) = \delta(x-y) (\partial_\mu \partial_\nu - \partial_\alpha^2 \delta_{\mu\nu})$$

and since $K_{\mu\nu} \partial_\nu \theta = 0$ we have a zero mode and $K_{\mu\nu}$ is not invertible.

Let us consider the example of a simple 2d integral ($b > 0$)

$$\mathcal{Z} = \int dx_1 dx_2 e^{-\frac{b}{2}(x_1 - x_2)^2} = \int d^2x e^{-\frac{1}{2} \vec{x}^T H \vec{x}} \quad \text{with} \quad H = b \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The integrand is invariant under $x_i \rightarrow x_i + \theta$, and associated to this invariance we have

a zero mode for H : $H \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$. $\mathcal{Z}$ is thus formally divergent. Let us assume however to

be interested only to average values of "gauge invariant" observables, i.e. of observables

$O(\vec{x})$ invariant under $x_i \rightarrow x_i + \theta$, like $f(x_1 - x_2)$:

$$\langle O \rangle = \frac{\int d^2x O(\vec{x}) e^{-\frac{1}{2} \vec{x}^T H \vec{x}}}{\int d^2x e^{-\frac{1}{2} \vec{x}^T H \vec{x}}}$$

The procedure we adopt to make this average value well defined is the following:

- a) we decompose the integrals in integrals along the gauge orbits ($x_i \rightarrow x_i + \theta$) and integrals along directions orthogonal to the orbits
- b) we consider only the integrals along directions orthogonal to the orbits if the following two requirements are satisfied
 - 1) we can select a single representative for each orbit
 - 2) the "weight" of the orbits is constant

The typical way to select a value from gauge orbits is to fix the value of a non gauge invariant quantity, like e.g. $\frac{x_1 + x_2}{2}$. For this purpose we insert inside the integrals the identity

$$1 = \int dy \delta\left(\frac{x_1 + x_2}{2} - y\right). \quad \text{Considering the case of } \mathcal{Z} \text{ we have}$$

$$\mathcal{Z} = \int dy \int d^2x \delta\left(\frac{x_1 + x_2}{2} - y\right) e^{-\frac{1}{2} \vec{x}^T H \vec{x}}$$

We now note that the value of the integral $\int d^2x \delta\left(\frac{x_1 + x_2}{2} - y\right) e^{-\frac{1}{2} \vec{x}^T H \vec{x}}$ is independent

of y : if we use $x'_i = x_i + y$ (i.e. $\vec{x}' = \vec{x} + y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, gauge transformation) we have

indeed $\vec{x}^T H \vec{x} = \vec{x}'^T H \vec{x}'$, $d^2x = d^2x'$, and thus

$$\int d^2x \delta\left(\frac{x_1+x_2}{2} - y\right) e^{-\frac{1}{2}\vec{x}^T H \vec{x}} = \int d^2x' \delta\left(\frac{x'_1+x'_2}{2} - y\right) e^{-\frac{1}{2}\vec{x}'^T H \vec{x}'} \quad \text{independent of } y.$$

We can thus define

$$\hat{Z} = \frac{Z}{\int dy} = \int d^2x \delta\left(\frac{x_1+x_2}{2} - y\right) e^{-\frac{1}{2}\vec{x}^T H \vec{x}}$$

obviously divergent, this is meaningful only after some IR regularization has been introduced

we have just seen that the result is independent of the specific value of y adopted

The same manipulations can be carried out also in the numerator, as for $\phi \propto O(\vec{x})$ is gauge invariant. In general one can think of using

$$\hat{Z} = \int d^2x \delta(f(\vec{x}) - y) e^{-\frac{1}{2}\vec{x}^T H \vec{x}}$$

where f is a generic gauge fixing function which satisfies the constraints 1 and 2 above.

NOTE In QED, using f linear in A_μ , the constraints 1 and 2 are satisfied, since gauge transformations are linear. In the non-Abelian case neither of the two constraints will be satisfied.

In Euclidean QED we can follow the same strategy adopted in the toy model:

$$Z_{\text{QED}} = \int DA_\mu D(\bar{\psi}, \psi) \prod_x \delta(\partial_\mu A_\mu(x) - h(x)) e^{-S_E}, \quad S_E = \int d^Dx \mathcal{L}_{\text{QED}}^{(E)}, \quad \mathcal{L}_{\text{QED}}^{(E)} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}(\not{D} + m)\psi + ie \bar{\psi} \not{A} \psi$$

NOTE: when we previously introduced the path-integral we started from the Hamiltonian, here we are proceeding by analogy using the final result. For the complete story see WEINBERG, ZINN-SUSTIN, ITZYKSON-ZUBER.

Since Z_{QED} is independent of the specific value of $h(x)$ (just as Z in the 2d toy model was independent of y) we can integrate on $h(x)$ with a Gaussian weight (with $\xi > 0$ arbitrary)

$$\begin{aligned} Z_{\text{QED}} &= \int DA D(\bar{\psi}, \psi) \int dh \prod_x \delta(\partial_\mu A_\mu(x) - h(x)) e^{-S_{\text{QED}}} \exp\left\{-\frac{1}{2\xi} \int d^Dx h^2(x)\right\} = \\ &= \int DA D(\bar{\psi}, \psi) \exp\left\{-S_{\text{QED}} - \frac{1}{2\xi} \int d^Dx (\partial_\mu A_\mu)^2\right\} \end{aligned}$$

and we thus find the gauge-fixed Euclidean Lagrangian density

$$\mathcal{L}_{\text{QED GF}}^{(E)} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}(\not{D} + m)\psi + ie \bar{\psi} \not{A} \psi + \frac{1}{2\xi} (\partial_\mu A_\mu)^2$$

The gauge-fixing term is not gauge invariant. Consequences of gauge invariance of the original theory are encoded in the WARD-TAKAHASHI identities to be discussed later on.

Going back to Hinkowski, we have

$$-\int d^Dx_E (\partial_\mu A_\mu)^2 \longrightarrow -i \int d^Dx (\partial_\mu A^\mu)^2 \quad (\text{using } x^0 = -ix^4 \text{ and } A_0 = iA^4)$$

$$(\partial_\mu A^\mu = \partial_0 A_0 - \partial_j A_j \rightarrow -\frac{1}{i} \frac{\partial}{\partial x^4} iA_4 - \partial_j A_j = -\partial_\mu A^\mu{}^{(E)})$$

$$\text{we thus have } \mathcal{L}_{\text{QED GF}}^{(H)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \underbrace{\bar{\psi}(i\not{D} - m)\psi}_{\text{with } D_\mu = \partial_\mu + ieA_\mu} - e \bar{\psi} \not{A} \psi - \frac{1}{2\xi} (\partial_\mu A^\mu)^2.$$

$$\mathcal{Z}[\mathcal{S}_\mu, \bar{\eta}, \eta] = \int \mathcal{D}A \mathcal{D}(\bar{\psi}, \psi) \exp \left\{ i \int d^D x \left[\mathcal{L}_{\text{QEDGF}}^{(H)} + \mathcal{S}_\mu A^\mu + \bar{\eta} \psi + \bar{\psi} \eta \right] \right\}$$

$$\begin{aligned} & \langle 0 | T \psi(x_1) \bar{\psi}(y_1) \dots \psi(x_m) \bar{\psi}(y_m) A_{\mu_1}(z_1) \dots A_{\mu_m}(z_m) | 0 \rangle = \\ & = \frac{(-i)^{2m+m}}{z[0]} \frac{\delta^{2m+m} z[\mathcal{J}, \bar{\eta}, \eta]}{\delta \bar{\eta}(x_1) (-\delta \eta(y_1)) \dots \delta \bar{\eta}(x_m) (-\delta \eta(y_m)) \delta \mathcal{J}^{\mu_1}(z_1) \dots \delta \mathcal{J}^{\mu_m}(z_m)} \Big|_{\mathcal{J}=\bar{\eta}=\eta=0} \end{aligned}$$

$\mathcal{L}_I^{(H)} = -e \bar{\Psi} A \Psi$ hence we have

$$Z[\mathcal{J}, \bar{\eta}, \eta] = \exp \left\{ i \int d^4x \, \mathcal{L}_I(\psi(x) \rightarrow \frac{\delta}{i\delta \bar{\eta}(x)}, \bar{\psi}(x) \rightarrow -\frac{\delta}{i\delta \eta(x)}, A_\mu(x) \rightarrow \frac{\delta}{i\delta \mathcal{J}^\mu(x)}) \right\} Z_{free}[\mathcal{J}, \bar{\eta}, \eta]$$

where $\mathcal{L}_{\text{free}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \bar{\psi} (i \not{\partial} - m) \psi$ and $\mathcal{Z}_{\text{free}}[\mathcal{J}, \bar{\eta}, \eta] = \mathcal{Z}_{\text{free}}^{(A)}[\mathcal{J}] \mathcal{Z}_{\text{free}}^{(\psi)}[\bar{\eta}, \eta]$.

We have seen before that

$$Z_{free}^{(4)}[\bar{\eta}, \eta] = Z_{free}^{(4)}[0] \exp \left\{ -i \int \bar{\eta}(x) \left(\frac{1}{i\partial^2 - m} \right)_{xy} \eta(y) d^2x d^2y \right\}$$

and we are now going to compute Z_{free}^A .

$$\begin{aligned}
 -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^4x &= -\frac{1}{2} \int \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) d^4x = \frac{1}{2} \int A_\nu (\partial_\alpha \partial^\alpha \delta^\nu_\mu - \partial_\mu \partial^\nu) A^\mu d^4x = \\
 &= \frac{1}{2} \int A^\mu(x) \delta(x-y) (\partial_\alpha \partial^\alpha \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu(y) d^4x d^4y
 \end{aligned}$$

$$-\frac{1}{2} \int (\partial_\mu A^\mu)^2 d^D x = -\frac{1}{2} \int (\partial_\mu A^\mu)(\partial_\nu A^\nu) d^D x = \frac{1}{2} \int A^\mu(x) \delta(x-y) \partial_\mu \partial_\nu A^\nu(y) d^D x d^D y$$

hence

$$S_{free}^{(A)} = \frac{1}{2} \int A_\mu(x) K^{\mu\nu}(x,y) A_\nu(y) d^D x d^D y \quad \text{with} \quad K^{\mu\nu}(x,y) = \delta(x-y) \left(\partial_\alpha \partial^\alpha \eta^{\mu\nu} - (1 - \frac{1}{\xi}) \partial^\mu \partial^\nu \right)$$

end

$$\begin{aligned} Z_{free}^A[J] &= \int DA \exp \left\{ \frac{i}{2} \int A_\mu(x) K^{\mu\nu}(x, y) A_\nu(y) d^Dx d^Dy + i \int J_\mu(x) A^\mu(x) d^Dx \right\} = \\ &= Z_{free}^A[0] \exp \left\{ -\frac{i}{2} \int J^\mu(x) K_{\mu\nu}^{-1}(x, y) J^\nu(y) d^Dx d^Dy \right\}. \end{aligned}$$

The fermion propagator is (the arrow goes from $\bar{\psi}$ to ψ , which is the direction of propagation)

of particles

$$\begin{array}{c} \bullet \xleftarrow{\quad} \bullet \\ \beta, y \qquad \alpha, x \end{array} = \langle 0 | T \psi_\beta(y) \bar{\psi}_\alpha(x) | 0 \rangle = \frac{(-i)^2}{2 \int_{\text{free}}(0)} \frac{\delta^2 Z_{\text{free}}^{(4)}[\bar{\eta}, \eta]}{\delta \bar{\eta}_\beta(y) (-\delta \eta_\alpha(x))} \Big|_{\bar{\eta}=\eta=0} =$$

$$= i \left(\frac{1}{i \not{p} - m} \right)_{yx} = i \int \frac{d^D k}{(2\pi)^D} \left(\frac{1}{k - m + i\epsilon} \right)_{\beta\alpha} e^{-ik(y-x)} = i \int \frac{d^D k}{(2\pi)^D} \frac{(k+m)_{\beta\alpha}}{k^2 - m^2 + i\epsilon} e^{-ik(y-x)} = i S_{\beta\alpha}(y-x)$$

The photon propagator is $\text{---}\overset{\nu\mu}{\bullet}\text{---} = \langle 0 | T A_\nu(y) A_\mu(x) | 0 \rangle = \frac{(-i)^2}{Z_{\text{free}}^{(A)}[0]} \frac{\delta^2 Z_{\text{free}}^{(A)}[J]}{\delta J^\nu(y) \delta J^\mu(x)} \Big|_{J=0}$

$$= i \, K_{\nu\mu}^{-1}(y, x)$$

with $\int d^D y K^{\alpha\mu}(x,y) K^{-1}_{\mu\beta}(y,z) = \delta^\alpha_\beta \delta(x-z)$.

If we define $K_{\mu\beta}^{-1}(y, z) = \int \tilde{D}_{\mu\beta}(p) e^{-ip(y-z)} \frac{d^D p}{(2\pi)^D}$ we have

$$\delta_\beta^\alpha \delta(x-z) = \int d^D y \delta(x-y) (-p^2 \eta^{\alpha\mu} + (1-\frac{1}{3}) p^\alpha p^\mu) e^{-ip(y-z)} \tilde{D}_{\mu\beta}(p) \frac{d^D p}{(2\pi)^D}$$

hence $\delta_\beta^\alpha = (-p^2 \eta^{\alpha\mu} + (1-\frac{1}{3}) p^\alpha p^\mu) \tilde{D}_{\mu\beta}(p)$. Let us introduce the ansatz

$$\tilde{D}_{\mu\beta}(p) = \frac{1}{p^2} (A \eta_{\mu\beta} + B p_\mu p_\beta / p^2)$$

$$\delta_\beta^\alpha = (-p^2 \eta^{\alpha\mu} + (1-\frac{1}{3}) p^\alpha p^\mu) \tilde{D}_{\mu\beta}(p) = \frac{1}{p^2} (-A p^2 \delta_\beta^\alpha + A (1-\frac{1}{3}) p^\alpha p_\beta - B p^\alpha p_\beta + B (1-\frac{1}{3}) p^\alpha p_\beta)$$

hence $A = -1$ and $-(1-\frac{1}{3}) - B + B(1-\frac{1}{3}) = 0$, thus $-\frac{B}{3} = 1 - \frac{1}{3}$ and $B = 1 - \frac{1}{3}$ and finally

$$\tilde{D}_{\mu\beta}(p) = -\frac{1}{p^2} (\eta_{\mu\beta} - (1-\frac{1}{3}) p_\mu p_\beta / p^2).$$

The photon propagator is thus given by

$$\langle 0 | T A_\nu(y) A_\mu(x) | 0 \rangle = i \int \tilde{D}_{\nu\mu}(p) e^{-ip(y-x)} \frac{d^D p}{(2\pi)^D} = -i \int \frac{\eta_{\nu\mu} - (1-\frac{1}{3}) p_\nu p_\mu / p^2}{p^2 + i\epsilon} e^{-ip(y-x)} \frac{d^D p}{(2\pi)^D} = i D_{\nu\mu}(y-x)$$

↑ seen by continuation from the Euclidean theory IR regularized with a photon mass.

The interaction Lagrangian $S_I^{(H)} = -\int d^D x \bar{\psi} \not{A} \psi$

generates the vertex

$$\begin{array}{c} \mu \\ \uparrow \\ x \text{---} \text{---} \text{---} \beta \\ \alpha \end{array} = -ie(\gamma^\mu)_{\beta\alpha} \int d^D x$$

Feynman rules in Euclidean space: we saw previously that the Euclidean propagator for Euclidean Dirac fermions is

$$\begin{array}{c} \text{---} \leftarrow \text{---} \\ \beta y \end{array} \begin{array}{c} \text{---} \rightarrow \text{---} \\ \alpha x \end{array} = \langle \psi_\beta(y) \bar{\psi}_\alpha(x) \rangle = \int \frac{d^D p}{(2\pi)^D} \frac{1}{-ip\!\!\!/ + m} e^{-ip(y-x)} = \int \frac{d^D p}{(2\pi)^D} \frac{(ip\!\!\!/ + m)_{\beta\alpha}}{p^2 + m^2} e^{-ip(y-x)}$$

The Euclidean free photon action is $\int (\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2) d^D x = \frac{1}{2} \int A_\mu(x) K_{\mu\nu}^{(E)}(x, y) A_\nu(y) d^D x d^D y$

with $K_{\mu\nu}^{(E)}(x, y) = \delta(x-y) [(1-\frac{1}{3}) \partial_\mu \partial_\nu - \partial_\alpha^2 \delta_{\mu\nu}]$ moreover

$$\int dA \exp \left\{ -\frac{1}{2} \int A_\mu K_{\mu\nu}^{(E)} A_\nu + \int A_\mu \tilde{J}_\mu \right\} \propto \exp \left\{ \frac{1}{2} \int \tilde{J}_\mu [K^{(E)}]_{\mu\nu}^{-1} \tilde{J}_\nu \right\} \text{ hence}$$

$$\text{anna}_{\mu\nu} = \langle A_\mu(y) A_\nu(x) \rangle = \int \frac{d^D k}{(2\pi)^D} \frac{\delta_{\mu\nu} - (1-\frac{1}{3}) K_\mu k_\nu / k^2}{k^2} e^{-ik(y-x)}$$

Finally the interaction action is $S_I^{(E)} = ie \int d^D x \bar{\psi} \not{A} \psi$ hence the vertex is

$$\begin{array}{c} \mu \\ \uparrow \\ x \text{---} \text{---} \text{---} \beta \\ \alpha \end{array} = -ie(\gamma^\mu)_{\beta\alpha} \int d^D x \quad \text{as in Minkowsky signature.}$$

NOTE: since in QED the three lines of the vertex are different, all symmetry factors are trivial.

The $\frac{1}{n!}$ of the Dyson series cancels the $n!$ coming from the permutations of the vertices and no further combinatoric factors are present.

In uncomputed diagrams needed for the computation of S-matrix elements external legs are associated to the wave functions of ingoing/outgoing particles: from the free field expressions

$$\psi_{free}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left\{ a_s(\vec{p}) u_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + b_s^\dagger(\vec{p}) \bar{v}_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right\}$$

$$\bar{\psi}_{free}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left\{ b_s(\vec{p}) \bar{v}_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + a_s^\dagger(\vec{p}) \bar{u}_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right\}$$

we get (using the orthogonality relations $\bar{u}_\alpha(\vec{p}) \gamma^0 u_\beta(\vec{p}) = 2E_p \delta_{\alpha\beta}$, $\bar{v}_\alpha(\vec{p}) \gamma^0 v_\beta(\vec{p}) = 2E_p \delta_{\alpha\beta}$, $\bar{u}_\alpha(\vec{p}) \gamma^0 v_\beta(-\vec{p}) = 0$, $\bar{v}_\alpha(\vec{p}) \gamma^0 u_\beta(-\vec{p}) = 0$)

$$a_s(\vec{k}) = \frac{1}{\sqrt{2E_p}} \int d^3x \bar{u}_s(\vec{k}) \gamma^0 \psi_{free}(\vec{x}) e^{i\vec{k}\cdot\vec{x}} ; \quad a_s^\dagger(\vec{k}) = \frac{1}{\sqrt{2E_p}} \int d^3x \bar{\psi}_{free}(\vec{x}) \gamma^0 u_s(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$$

$$b_s(\vec{k}) = \frac{1}{\sqrt{2E_p}} \int d^3x \bar{\psi}_{free}(\vec{x}) \gamma^0 v_s(\vec{k}) e^{i\vec{k}\cdot\vec{x}} ; \quad b_s^\dagger(\vec{k}) = \frac{1}{\sqrt{2E_p}} \int d^3x \bar{v}_s(\vec{k}) \gamma^0 \psi_{free}(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}$$

Using these expressions we can get the LSZ reduction formulae for fermions (see, e.g. ITZYKSON, ZUBER "Quantum field theory" § 5.1.6) and we find the following correspondences for wave functions:

ingoing fermion: correlation function with $\bar{\psi}$, wave function $u_\alpha(\vec{k})$

outgoing " : " " ψ , $\bar{u}_\alpha(\vec{k})$

ingoing antifermion: correlation function with ψ , wave function $\bar{v}_\alpha(\vec{k})$

outgoing " : " " $\bar{\psi}$, " $v_\alpha(\vec{k})$

For photons we have analogously

ingoing photon: $\epsilon_\mu^{(\lambda)}(\vec{k})$

outgoing photon: $\epsilon_\mu^{(\lambda)}(\vec{k})^*$

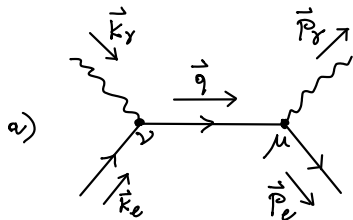
The integrals on the coordinates that are associated to vertices, together with exponentials of propagators in coordinate space, give the $(2\pi)^D \delta(\sum p_{in} - \sum p_{out})$ enforcing momentum conservation.

EXAMPLES

Let us consider the process

$$e^- + \gamma \rightarrow e^- + \gamma$$

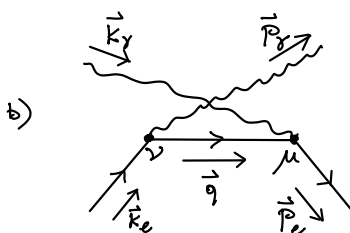
$$\vec{k}_e, \sigma_e; \quad \vec{k}_\gamma, \lambda_\gamma; \quad \vec{p}_e, \sigma_f; \quad \vec{p}_\gamma, \lambda_f$$



$$\bar{u}_{\sigma_f}(\vec{p}_e) (-ie\gamma^\mu) i \frac{\not{q} + m}{q^2 - m^2} (-ie\gamma^\nu) u_{\sigma_e}(\vec{k}_e) \underbrace{\epsilon_\mu^{(\lambda_f)*}(\vec{p}_\gamma) \epsilon_\nu^{(\lambda_\gamma)}(\vec{k}_\gamma)}$$

$$\text{with } q = k_e + k_\gamma = p_e + p_\gamma$$

note the different indices!



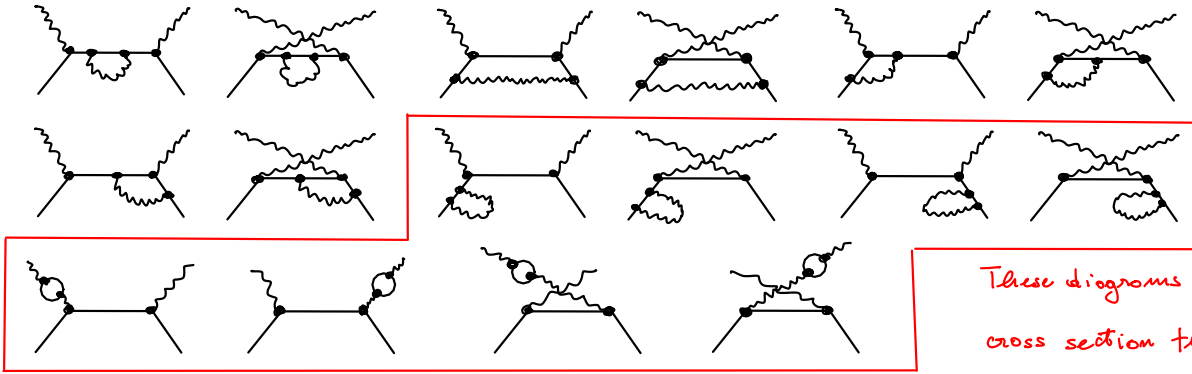
$$\bar{u}_{\sigma_f}(\vec{p}_e) (-ie\gamma^\mu) i \frac{\not{q} + m}{q^2 - m^2} (-ie\gamma^\nu) u_{\sigma_e}(\vec{k}_e) \underbrace{\epsilon_\nu^{(\lambda_f)*}(\vec{p}_\gamma) \epsilon_\mu^{(\lambda_\gamma)}(\vec{k}_\gamma)}$$

$$\text{with } q = k_e - p_\gamma = p_e - k_\gamma$$

The S-matrix element of the process is $(2\pi)^4 \delta(k_e + k_\gamma - p_e - p_\gamma) [(a) + (b)]$ and the term in square brackets is often denoted by $\mathcal{M}$. ↗ 4-vector!

$$d\sigma = \frac{1}{4I} |\mathcal{M}|^2 d\phi^{(n)}, \text{ where } I = \sqrt{(k_{1\mu} k_2^\mu) - m_1^2 m_2^2} \text{ and } d\phi^{(n)} = (2\pi)^4 \delta(\sum p_j - \sum p_i) \prod_{j=1}^n \frac{d^3 p_j}{(2\pi)^3 2p_j^0}$$

Diagrams contributing at the next to leading order are



These diagrams only enters the cross section through the $\mathcal{M}$ entering LSZ formulae

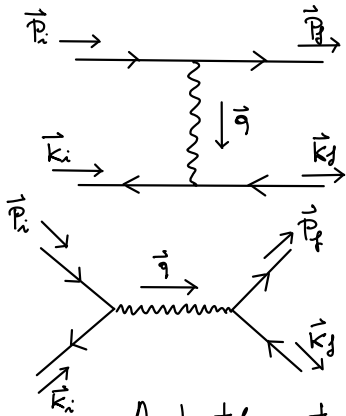
3n diagrams with loops the δ generated by vertices are

not enough to fix all momenta, and integrals on momenta remains in the amplitude

$$\int \bar{u}_{\alpha_j}(\vec{p}_e) (-ie\gamma_\mu) i \frac{\not{q}_1 + m}{q_1^2 - m^2} (-ie\gamma^\alpha) i \frac{\not{q}_2 + m}{q_2^2 - m^2} (-ie\gamma^\beta) i \frac{\not{q}_3 + m}{q_3^2 - m^2} u_{\alpha_i}(\vec{k}_e) \cdot \epsilon_\nu^{(\lambda_i)}(\vec{k}_\gamma) \epsilon_\mu^{(\lambda_j)*}(\vec{p}_\gamma) (-i) \frac{\eta_{\alpha\beta} - (1-\xi) \frac{k_\alpha k_\beta}{k^2}}{k^2} \frac{d^4 k}{(2\pi)^4}$$

where $k_\gamma + k_e = q_3 = q_2 + k = q_1 = p_\gamma + p_e$ hence
$$\begin{cases} q_1 = p_\gamma + p_e = k_\gamma + k_e \\ q_2 = q_1 - k \\ q_3 = q_1 \end{cases}$$

Let us consider the process $e^+ + e^- \longrightarrow e^+ + e^-$
 $\vec{p}_i, s_i \quad \vec{k}_i, \sigma_i \quad \vec{p}_f, s_f \quad \vec{k}_f, \sigma_f$



$$\bar{u}_{s_f}(\vec{p}_f) (-ie\gamma^\mu) u_{s_i}(\vec{p}_i) \bar{u}_{\sigma_i}(\vec{k}_i) (-ie\gamma^\nu) u_{\sigma_f}(\vec{k}_f) (-i) \frac{\eta_{\mu\nu} - (1-\xi) q_\mu q_\nu / q^2}{q^2}$$

with $q = p_i - p_f$

$$\bar{u}_{\sigma_i}(\vec{k}_i) (-ie\gamma^\mu) u_{s_i}(\vec{p}_i) \bar{u}_{s_f}(\vec{p}_f) (-ie\gamma^\nu) u_{\sigma_f}(\vec{k}_f) (-i) \frac{\eta_{\mu\nu} - (1-\xi) q_\mu q_\nu / q^2}{q^2}$$

with $q = p_i + k_i$

And the two diagrams have a relative minus sign due to Wick theorem (Fermions!)

$$\begin{aligned} & \begin{array}{c} 1 \rightarrow 3 \rightarrow 5 \\ 2 \leftarrow 4 \leftarrow 6 \end{array} \quad \langle \psi_2 \bar{\psi}_4 \rangle \langle \psi_4 \bar{\psi}_6 \rangle \langle \psi_5 \bar{\psi}_3 \rangle \langle \psi_3 \bar{\psi}_1 \rangle \\ & \begin{array}{c} 1 \rightarrow 3 \rightarrow 5 \\ 2 \leftarrow 4 \leftarrow 6 \end{array} \quad \langle \psi_5 \bar{\psi}_4 \rangle \langle \psi_4 \bar{\psi}_6 \rangle \langle \psi_2 \bar{\psi}_3 \rangle \langle \psi_3 \bar{\psi}_1 \rangle \end{aligned}$$

second index 4631 in both the cases

first index 2453 and 5423

1 exchange $\Rightarrow$ a different sign.

RENORMALIZATION

r1

Momentum space conventions

We previously defined

$$W[\mathcal{J}] = \sum \frac{1}{m!} \int W_m(x_1, \dots, x_m) \prod_{i=1}^m d^D x_i \mathcal{J}(x_i) \quad , \quad T[\varphi] = \sum \frac{1}{m!} \int T_m(x_1, \dots, x_m) \prod_{i=1}^m d^D x_i \varphi(x_i)$$

where we assumed for the sake of the simplicity that $\langle \phi(x) \rangle = 0$, otherwise we have to use $\varphi(x) \rightarrow \eta(x) = \varphi(x) - \langle \phi(x) \rangle$ in the second expression, see PSF11.

In translation invariant theories it is convenient to use the momentum space fields

$$\varphi(x) = \int e^{ipx} \tilde{\varphi}(p) d^D p \quad , \quad \mathcal{J}(x) = \int e^{ipx} \tilde{\mathcal{J}}(p) d^D p \quad \text{NOTE: ingoing momenta.}$$

and to introduce the Fourier transforms of W_m and T_m by

$$(2\pi)^D \delta\left(\sum_k p_k\right) \tilde{W}_m(p_1, \dots, p_m) = \int W_m(x_1, \dots, x_m) \prod_{i=1}^m d^D x_i e^{i x_k p_k}$$

where the δ is due to the fact that $W_m(x_1, \dots, x_m)$ depends only on $x_k - x_1$, hence we can change variable $x_1 \rightarrow y_1 = x_1$ and $x_k \rightarrow y_k = x_k - x_1$ (for $k > 1$) and by integrating on y_1 we get the δ . So

$$\begin{aligned} W[\mathcal{J}] &= \sum_m \frac{1}{m!} \int W_m(x_1, \dots, x_m) \prod_{i=1}^m d^D x_i \int e^{i p_i x_i} \tilde{\mathcal{J}}(p_i) d^D p_i = \\ &= \sum_m \frac{1}{m!} \int (2\pi)^D \delta\left(\sum p_k\right) \tilde{W}_m(p_1, \dots, p_m) \prod_{i=1}^m d^D p_i \tilde{\mathcal{J}}(p_i) \end{aligned}$$

and the inverse transformation is

$$W_m(x_1, \dots, x_m) = \int (2\pi)^D \delta\left(\sum p_k\right) \tilde{W}_m(p_1, \dots, p_m) \prod_{i=1}^m \frac{d^D p_i}{(2\pi)^D} e^{-i p_i x_i}$$

Analogous manipulations can be used for T .

In particular we have

$$W_2(x, y) = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} e^{-i p_1 x - i p_2 y} (2\pi)^D \delta(p_1 + p_2) \tilde{W}_2(p_1, p_2) = \int \frac{d^D p_2}{(2\pi)^D} e^{i p_2 (x - y)} \tilde{W}_2(-p_2, p_2)$$

and analogously for T_2 . In the following, for simplicity, we will write just $\tilde{W}_2(p)$ instead of $\tilde{W}_2(-p, p)$, hence $\tilde{W}_2(p) = \frac{1}{p^2 + m^2}$ in the free scalar theory, and

$$\int T_2(x, y) W_2(y, z) d^D y = \delta(x - z) \quad \text{becomes} \quad \tilde{T}_2(p) \tilde{W}_2(p) = 1 \quad ,$$

while the Fourier transform of the amputated connected functions are

$$\tilde{W}_{m, \text{amp}}(p_1, \dots, p_m) = \tilde{W}_m(p_1, \dots, p_m) \prod_{i=1}^m \left(\tilde{W}_2(p_i) \right)^{-1} \quad ,$$

and for $\tilde{T}_m$ with $m \geq 2$ we have

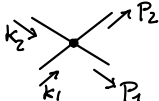
$$\tilde{T}_3(p_1, p_2, p_3) = - \tilde{W}_{3, \text{amp}}(p_1, p_2, p_3)$$

$$\begin{aligned} \tilde{T}_4(p_1, p_2, p_3, p_4) &= - \tilde{W}_{4, \text{amp}}(p_1, p_2, p_3, p_4) + \tilde{W}_{3, \text{amp}}(p_1, p_2, -p_1 - p_2) \tilde{W}_2(p_1 + p_2) \tilde{W}_{3, \text{amp}}(p_3, p_4, -p_3 - p_4) \\ &\quad + \text{crossed} \end{aligned}$$

as can be seen by Fourier transforming the relation at page PSF12

The logic of renormalization

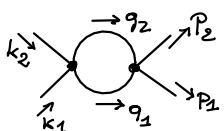
Let us consider the amputated 4 point of the Euclidean ϕ^4 theory $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$, which when continued to Minkowski it is related to the $2 \rightarrow 2$ scattering process.

• tree level  contribution to $\tilde{W}_{4,amp}$: $-g$

The 4 point function is $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = -g \int d^D y \Delta_E(x_1, y) \Delta_E(x_2, y) \Delta_E(x_3, y) \Delta_E(x_4, y)$

and $\Delta_E(x, y) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} e^{-iq(x-y)}$ hence we find

$-g \int \prod_{i=1}^4 \frac{d^D q_i}{(2\pi)^D} \frac{1}{q_i^2 + m^2} e^{-iq_i x_i} (2\pi)^D \delta(\sum_i q_i)$ which after F.T. and amputation reduces to $-g (2\pi)^D \delta(\sum q_i)$, hence $\tilde{W}_{4,amp} = -g$ at tree level

• 1 loop  + crossed diagrams (symmetry factor $\frac{1}{2}$)

contribution to $\tilde{W}_{4,amp}$: $\frac{1}{2} g^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2) [(k_1 + k_2 + q)^2 + m^2]} + \text{crossed}$

each vertex gives a $-g (2\pi)^D \delta(\sum q_i)$ by F.T. and amputating we find

$$\frac{1}{2} g^2 \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{1}{q_1^2 + m^2} \frac{1}{q_2^2 + m^2} (2\pi)^D \delta(k_1 + k_2 - q_1 - q_2) (2\pi)^D \delta(q_1 + q_2 - p_1 - p_2)$$

which by the change of variable $q_1, q_2 \rightarrow q_1, q = q_1 + q_2$ reduces to

$$\begin{aligned} \frac{1}{2} g^2 \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{q_1^2 + m^2} \frac{1}{(q - q_1)^2 + m^2} (2\pi)^D \delta(k_1 + k_2 - q) (2\pi)^D \delta(q - p_1 - p_2) \\ = \frac{1}{2} g^2 (2\pi)^D \delta(k_1 + k_2 - p_1 - p_2) \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2 + m^2) [(k_1 + k_2 - q_1)^2 + m^2]} \end{aligned}$$

This integral is UV divergent for $D \geq 4$, since for large q it behaves as $\int d^D q / q^4$, while it is IR convergent in all the cases as far as $m \neq 0$. If we introduce $k = k_1 + k_2$ we have

$$\frac{1}{(k+q)^2 + m^2} = \frac{1}{q^2 + 2kq + k^2 + m^2} = \frac{1}{(q^2 + m^2)(1 + \frac{k^2 + 2kq}{q^2 + m^2})} \simeq \frac{1}{q^2 + m^2} \left(1 - \underbrace{\frac{k^2 + 2kq}{q^2 + m^2}}_{O(1/q) \text{ for large } q} \right)$$

and we see that the divergence affects only the zero momentum part of the integral. It can thus be eliminated by subtracting $\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)^2}$ to the previous expression, which corresponds to adding a vertex  $= A (2\pi)^D \delta(\sum q_i)$ (in momentum space) or $A \int \phi^4(x) d^D x$ (in coordinate space) with divergent A , which can be written as $\Delta \mathcal{L} = \frac{\delta g}{4!} \phi^4(x)$ and formally absorbed in g .

To investigate this point in a systematic way let us consider the form of the 1-loop

effective action obtained before (see PSF 16)

$$\Gamma^{(1)}[\varphi] = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} g^n \int dx_1 \dots dx_n \frac{\varphi^2(x_1)}{2} \Delta_E(x_1, x_2) \frac{\varphi^2(x_2)}{2} \Delta_E(x_2, x_3) \dots \frac{\varphi^2(x_n)}{2} \Delta_E(x_n, x_1)$$

while the tree level contribution is $\Gamma^{(0)}[\varphi] = S[\varphi]$, hence in particular

$$\tilde{\Gamma}_2^{(0)}(k) = k^2 + m^2, \quad \tilde{\Gamma}_4^{(0)}(k) = g, \quad \tilde{\Gamma}_n^{(0)} = 0 \text{ if } n > 4$$

$n=1$) we have

$$\frac{1}{2} g \int dx_1 \frac{\varphi^2(x_1)}{2} \Delta_E(x_1, x_1) \quad \text{and using } \Delta_E(x, y) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} e^{ik(x-y)}$$

and $\varphi(x) = \int \tilde{\varphi}(p) e^{ipx} d^D p$ we get

$$\frac{1}{4} g \int d^D x_1 d^D p_1 d^D p_2 e^{i(p_1 + p_2)x_1} \tilde{\varphi}(p_1) \tilde{\varphi}(p_2) \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = \frac{1}{4} g \int d^D p_1 d^D p_2 (2\pi)^D \delta(p_1 + p_2) \tilde{\varphi}(p_1) \tilde{\varphi}(p_2) \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2}$$

$$\text{thus we found } \tilde{\Gamma}_2^{(1)} = \frac{1}{2} g \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2}$$

Since $\tilde{\Gamma}_2 = k + \Sigma$ (k = quadratic kernel, not momentum!) we have $k_2 = (k + \Sigma)^{-1} = k^{-1} (1 + \Sigma k)^{-1} =$

$= k^{-1} - k^{-1} \Sigma k^{-1} + \dots$ and the - sign of the second term correct the - sign of the vertex

$-g$ in perturbation theory, thus the result we just obtained from the formal expression of $\Gamma^{(1)}$

corresponds to , which has symmetry factor $1/2$.

$n=2$) we have

$$- \frac{1}{2^4} g^2 \int d^D x_1 d^D x_2 \varphi^2(x_1) \Delta_E(x_1, x_2) \varphi^2(x_2) \Delta_E(x_2, x_1) =$$

$$= - \frac{1}{2^4} g^2 \int d^D x_1 d^D x_2 d^D p_1 d^D p_2 d^D p_3 d^D p_4 e^{ix_1(p_1 + p_2)} e^{ix_2(p_3 + p_4)} \tilde{\varphi}(p_1) \tilde{\varphi}(p_2) \tilde{\varphi}(p_3) \tilde{\varphi}(p_4) \cdot$$

$$\cdot \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2 + m^2} e^{ik_1(x_1 - x_2)} \int \frac{d^D k_2}{(2\pi)^D} \frac{1}{k_2^2 + m^2} e^{ik_2(x_2 - x_1)} =$$

$$= - \frac{1}{2^4} g^2 \int \left(\prod_i d^D p_i \right) \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{k_1^2 + m^2} \frac{1}{k_2^2 + m^2} (2\pi)^D \delta(p_1 + p_2 + k_1 - k_2) (2\pi)^D \delta(p_3 + p_4 - k_1 + k_2) \prod_i \tilde{\varphi}(p_i) =$$

$$= - \frac{1}{2^4} g^2 \int \left(\prod_i d^D p_i \right) (2\pi)^D \delta(\sum p_i) \left(\prod_i \tilde{\varphi}(p_i) \right) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2 + m^2} \frac{1}{(p_1 + p_2 + k_1)^2 + m^2}$$

hence

$$\tilde{\Gamma}_4^{(1)} = - \frac{4!}{2^4} g^2 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2 + m^2} \frac{1}{(p_1 + p_2 + k_1)^2 + m^2}$$

NOTE: the fact that there seems to be no dependence on p_3, p_4 is only an artifact of the way in which things are written, since in the integral which define Γ we sum on everything.

$$- \frac{4!}{2^4} = - \frac{3}{2} \quad \text{and this result can also be obtained from the graphs } \times \bigcirc \times + \text{crossed (3 total diagrams):}$$

the minus sign is due to the relation between $\tilde{\Gamma}_n$ and $\mathcal{W}_n$, the graphs have symmetry

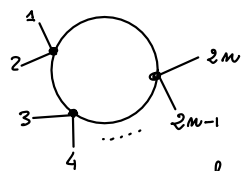
factors $\frac{1}{2}$, and in the effective action we sum on everything so we can rename

the momenta.

generic m) for generic m it is possible to obtain

$$\tilde{\Gamma}_{2m}^{(1)} = -(-g)^m \frac{(2m-1)!}{2^m} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} \cdot \frac{1}{(k+p_1+p_2)^2 + m^2} \cdots \frac{1}{(k+p_{2m-3}+p_{2m-2})^2 + m^2}$$

which corresponds to



which has symmetry factor

$$\frac{1}{m!} \left(\frac{1}{4!}\right)^m m! (4 \cdot 3)^m \frac{(2m-1)!}{2^m} = \frac{(2m-1)!}{2^m}$$

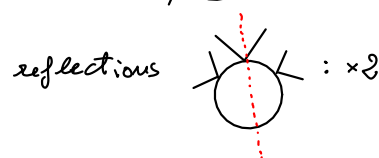
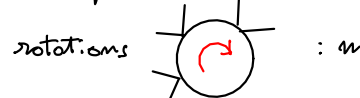
from the
Taylor series
of exp

from
 m vertices

ways to
connect the
vertices to create
a loop:

$$4m \cdot 4 \cdot (m-1) \cdot 3 \cdot 4(m-2) \cdots 4 \cdot 3 \cdot 3$$

ways to assign $2m$ momenta: $(2m)!$ permutations
but $2m$ are equivalent due to



We thus have, in summary

$\tilde{\Gamma}_2^{(1)}$ is divergent if $D \geq 2$

$\tilde{\Gamma}_4^{(1)}$ is divergent if $D \geq 4$

$\tilde{\Gamma}_{2m}^{(1)}$ is divergent if $D \geq 2m$

Let us consider for simplicity just the case $D=4$, in which only $\tilde{\Gamma}_2^{(1)}$ and $\tilde{\Gamma}_4^{(1)}$ diverge.

If we regularize in some way the integrals, e.g. with a cut-off on momenta (more details on regularizations will be discussed later on), we can isolate the divergent contributions of the Taylor expansion in the external momenta, thus obtaining the divergent part of $\Gamma^{(1)}$.

Using the Schwinger proper time regularization one gets (see ZINN-SUSTIN "Quantum field theory and critical phenomena" app. A9)

$$\Gamma_{\text{div}}^{(1)} = \frac{1}{32\pi^2} \left\{ \left(\frac{\Lambda^2}{2} g - g m^2 \log \frac{\Lambda}{m} \right) \int \psi^2(x) dx - \frac{g^2}{4} \log \frac{\Lambda}{m} \int \psi^4(x) dx \right\}$$

NOTE: the coefficients of the divergent terms typically depend on the regularization scheme adopted

The idea of the Schwinger proper time regularization: $\frac{1}{x} = \int_0^\infty e^{-xt} dt$ and

$$\log y = \int_1^y \frac{1}{x} dx = \int_1^y dx \int_0^\infty dt e^{-xt} = \int_0^\infty dt \left(\frac{1}{t} e^{-t} - \frac{1}{t} e^{-yt} \right) \text{ hence}$$

$$\log y - \log z = - \int_0^\infty \frac{1}{t} (e^{-zt} - e^{-yt}) dt \text{ and passing to (Hermitian & positive) operators}$$

$$\text{tr}(\log H - \log H_0) = - \int_0^\infty \frac{dt}{t} \text{tr}(e^{-tH} - e^{-tH_0}). \text{ Using this expression to rewrite the } \text{tr} \log \text{ of}$$

the effective action (see PSF 16) divergences appear from the $t \rightarrow 0$ integration region, hence to

regularize we use $\int_0^\infty dt \rightarrow \int_\epsilon^\infty dt$ with fixed $\epsilon > 0$, moreover $[t] = 1/[H]$ and $[H] = \left[\frac{\delta S}{\delta \psi \delta \bar{\psi}} \right] = E^2$

hence if we define $E = 1/\Lambda^2$ we have $[\Lambda] = E$

NOTE : in the ϕ^4 theory at 1-loop there is no divergent term proportional to $(\partial_\mu \phi)^2$, but this happens by chance. At 2-loop it appears, and in other models it is present already starting from 1-loop.

$\Gamma_{\text{div}}^{(1)}$ thus has the same form of the classical action S , but for divergent coefficients. If we consider the modified action $S(\phi) + \delta S_1(\phi)$, where

$$\delta S_1(\phi) = -b_2(\Lambda) \int d^4x \phi^2(x) - b_4(\Lambda) \int d^4x \phi^4(x)$$

and we fix the coefficients $b_i(\Lambda)$ in such a way that $\Gamma_{\text{div}}^{(1)} + \delta S_1$ is not divergent, then the corresponding 1-loop effective action $S(\phi) + \delta S_1(\phi) + \Gamma^{(1)}(\phi)$ is not divergent for $\Lambda \rightarrow \infty$.

NOTE δS_1 is already $\mathcal{O}(\hbar)$ hence to compute $\Gamma^{(1)}$ up to $\mathcal{O}(\hbar)$ we only need $S(\phi)$ and the result is the one quoted before.

In RENORMALIZABLE theories this procedure of absorbing divergences by introducing divergent BARE couplings can be carried out to any order in $\hbar$. The general idea is always that of introducing Λ -dependent bare parameters in such a way that correlation functions are Λ -independent order by order in $\hbar$ (i.e. in the number of loops).

Renormalizable theories are UV-insensitive : even if a physical cut-off were present (given by "new" physics) their large distance / small energy properties would be describable to arbitrary precision by a finite number of parameters.

1-loop divergences in QED

Feynman rules in Minkowski space : $\text{propagator} = -i \frac{\partial_{\mu\nu} - (1-\xi)k_\mu k_\nu / k^2}{k^2} = iD_{\mu\nu}(k)$

$$\text{fermion line} = \frac{i}{\not{k} - m + i\epsilon} = iS(k) \quad \text{vertex} = -ie \gamma^\mu_{ab}$$

Let us start by considering the photon two point function :

$$G_{\mu\nu}^{(AA)}(k) = \text{tree} + \text{1-loop} + \dots = iD_{\mu\nu}(k) + iD_{\mu\alpha} i\pi^{\alpha\beta}(k) iD_{\beta\nu}(k) + \dots$$

1-loop photon self-energy $i\pi^{\alpha\beta}(k) = - \int \frac{d^4q}{(2\pi)^4} \text{tr} \left(\frac{i}{\not{q} - m} (-ie\gamma^\alpha) \frac{i}{\not{q} - \not{k} - m} (-ie\gamma^\beta) \right)$

and in $D=4$ it seems to be quadratically divergent.

We now compute $K_\alpha \pi^{\alpha\beta}(k)$: using $-ie K_\alpha \gamma^\alpha = -ie [(\not{q} - m) - (\not{q} - \not{k} - m)]$ we have

$$\frac{1}{\not{q} - m} (-ie \not{k}) \frac{1}{\not{q} - \not{k} - m} = -ie \left(\frac{1}{\not{q} - \not{k} - m} - \frac{1}{\not{q} - m} \right)$$

using this identity we can write

$$ik_\alpha \Pi^{\alpha\beta}(k) = -e^2 \int \frac{d^D q}{(2\pi)^D} \left[t_2 \left(\frac{1}{q-k-m} \gamma^\beta \right) - t_2 \left(\frac{1}{q-m} \gamma^\beta \right) \right]$$

and if the regularization method adopted is consistent with a shift of the integration variables we get $k_\alpha \Pi^{\alpha\beta}(k) = 0$ by using $q \rightarrow q+k$ in the first term.

NOTE: the introduction of a simple momentum cut-off does not allow for a shift of the integration variables.

We thus have $\Pi^{\alpha\beta}(k) = (k^2 g^{\alpha\beta} - k^\alpha k^\beta) \Pi(k^2)$ and on dimensional ground $\Pi(k^2)$ is only logarithmically divergent, since we reduced by 2 the mass dimension of the integrand.

The relation $k_\alpha \Pi^{\alpha\beta}(k) = 0$ can be shown to be valid at all orders in $\hbar$ using the WARD-TAKAHASHI identities, where $i\Pi^{\alpha\beta}$ is the contribution to the photon two-point function of all connected, amputated and 1PI diagrams, so

$$G_{\mu\nu}^{(AA)} = \text{tree} + \text{1-loop} + \text{2-loop} + \dots$$

and since $i\Pi^{\alpha\beta}(k) iD_{\beta\nu}(k) = i(k^2 g^{\alpha\beta} - k^\alpha k^\beta) \Pi(k^2) (-i) \frac{\partial_{\beta\nu} - k_\beta k_\nu / k^2}{k^2} = \left(\delta^\alpha_\nu - \frac{k^\alpha k_\nu}{k^2} \right) \Pi(k^2)$,

with $\delta^\alpha_\nu - \frac{k^\alpha k_\nu}{k^2} = P^\alpha_\nu$ the projector on the directions perpendicular to k , we have

$$\begin{aligned} G_{\mu\nu}^{(AA)}(k) &= iD_{\mu\nu} + iD_{\mu\alpha} \underbrace{i\Pi^{\alpha\beta} iD_{\beta\nu}}_{P^\alpha_\nu \Pi} + iD_{\mu\alpha} \underbrace{i\Pi^{\alpha\beta} iD_{\beta\gamma}}_{P^\alpha_\gamma \Pi} \underbrace{i\Pi^{\gamma\delta} iD_{\delta\nu}}_{P^\delta_\nu \Pi} + \dots = iD_{\mu\nu} + iD_{\mu\alpha} P^\alpha_\nu (\Pi(k^2) + \Pi^2(k^2) + \dots) = \\ &= -i \frac{\partial_{\mu\nu} - (1-\xi) k_\mu k_\nu / k^2}{k^2} - i \frac{\partial_{\mu\nu} - k_\mu k_\nu / k^2}{k^2} (\Pi(k^2) + \Pi^2(k^2) + \dots) = \\ &= -i \xi \frac{k_\mu k_\nu}{k^4} - i \frac{\partial_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 (1 - \Pi(k^2))} \end{aligned}$$

hence the longitudinal component of the two point function does not renormalize.

If we consider the interaction of conserved currents ($\partial_\mu J^\mu = 0$) we have $k_\mu \tilde{J}^\mu(k) = 0$, moreover in general we do not expect poles at $k=0$ in $\Pi(k^2)$ (Higgs mechanism is an exception), hence we see that the photon remains massless even after renormalization. The factor $\frac{1}{1-\Pi(k^2)}$ however induce a (divergent) change of normalization at the pole, see the ξ s in the discussion of the Kallen-Lehman representation and LSZ reduction formulae.

We now consider the two point function of the electron

$$G^{\psi\bar{\psi}}(p) = \text{tree} + \text{1-loop} + \dots = \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m} (-i\Sigma(p)) \frac{i}{\not{p} - m} + \dots$$

$$-i\Sigma(p) = \int \frac{d^D q}{(2\pi)^D} (-ie\gamma^\mu) \frac{1}{\not{q} - m} (-ie\gamma^\nu) \frac{-i\partial_{\mu\nu}}{(q-p)^2} = \text{1-loop self-energy of the electron}$$

and for the sake of the simplicity we used $\xi=1$.

The exact dependence of the electron two point function on the gauge fixing parameter ξ is known but nontrivial (see ZINN-SUSTIN "Quantum theory of field and critical phenomena" § 18.7).

Since $\Sigma(p)$ a linear divergence is present (which reduces to log. divergence if the regulator is consistent with Lorentz invariance) and some care is needed to cope with an IR divergence, so it is convenient to introduce a photon mass in intermediate steps.

As in the photonic case let us denote by $-i\Sigma(p)$ the sum of connected amputated 1PI diagrams contributing to the electron two-point function (electron self-energy). We can then write

$$G^{\psi\bar{\psi}}(p) = \text{diagram series} = \frac{i}{\not{p} - m} \left(1 + \Sigma(p) \frac{1}{\not{p} - m} + \Sigma(p) \frac{1}{\not{p} - m} \Sigma(p) \frac{1}{\not{p} - m} + \dots \right) = \frac{i}{\not{p} - m - \Sigma(p)}$$

where in the last step we used the fact that $\Sigma(p)$ is function only of $\not{p}$ and thus commutes with $\not{p}$. If we call m' the position of the pole we have the implicit equation $m' = m + \Sigma(p)|_{\not{p}=m'}$ and close to the mass shell $\Sigma(p) \simeq \Sigma(p)|_{\not{p}=m'} + \underbrace{\frac{\partial \Sigma(p)}{\partial \not{p}}|_{\not{p}=m'}}_{1-A} \cdot (\not{p} - m')$ hence the denominator of the two point function is

$$\not{p} - m - \Sigma(p) \simeq \not{p} - m - (m' - m) + (A - 1)(\not{p} - m') = A(\not{p} - m')$$

hence $\Sigma(p)$ renormalize the electron mass and the normalization factor Z entering the Kallen-Lehman representation and LSZ reduction formulae ($Z = 1/A$).

Alternatively we can define m to be the physical mass corresponding to the pole, so when performing the renormalization we have to fix the constraint $\Sigma(p)|_{\not{p}=m} = 0$ and $\left(\frac{\partial \Sigma(p)}{\partial \not{p}}|_{\not{p}=m} \right)^{-1}$ gives the Z of the Kallen-Lehman representation.

Note that for scalar fields exactly the same happens: $\text{diagram} = \frac{i}{p^2 - m^2}$ and if we call $-i\Sigma(p^2)$ the sum of all amputated connected 1PI graphs we have

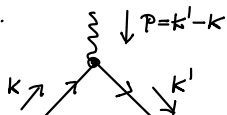
$$G^{(\phi\phi)}(p) = \text{diagram series} = \frac{i}{p^2 - m^2} \left(1 + \frac{1}{p^2 - m^2} \Sigma(p^2) + \frac{1}{p^2 - m^2} \Sigma(p^2) \frac{1}{p^2 - m^2} \Sigma(p^2) + \dots \right) = \frac{i}{p^2 - m^2 - \Sigma(p^2)}$$

and we can fix $\Sigma(p^2)|_{p^2=m^2} = 0$, $(1-Z)^{-1} = \frac{\partial \Sigma(p^2)}{\partial p^2}|_{p^2=m^2}$

Let us finally consider the vertex, i.e. the three point function

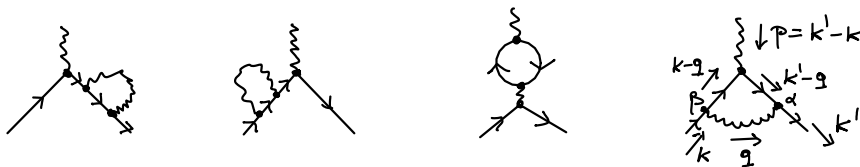
$$G_{\mu}^{\psi\psi A} = \langle 0 | T \psi(x) \bar{\psi}(y) A_{\mu}(z) | 0 \rangle.$$

At tree level only the diagram



is present, which corresponds

to $iS(k')(-ie\gamma^\nu)iS(k)iD_{\mu\nu}(p)$, while at one loop we have the diagrams



and only the last diagram is associated to a divergence we have not yet computed. The contribution of the last diagram is $iS(k')\Lambda^\nu(k',k)iS(k)iD_{\mu\nu}(p)$ where

$$\Lambda^\nu(k',k) = \int \frac{d^D q}{(2\pi)^D} (-ie\gamma_\alpha)iS(k'-q)(-ie\gamma^\nu)iS(k-q)(-ie\gamma_\beta)\frac{-ig^{\alpha\beta}}{q^2}$$

where we again used, for the sake of the simplicity, the gauge $\xi=1$ and a IR divergence is present, which requires the introduction of a photon mass (for cancellation of IR divergences in "physical" cross sections and jets see, e.g., WEINBERG "The quantum theory of fields" vol I, § 13, HUTA "Foundations of quantum chromodynamics" § 6). This integral is logarithmically divergent in $D=4$ and the ξ -dependence of Λ^ν is related to that of the fermion two points functions by the WARD-TAKAHASHI identities.

POWER COUNTING

QED in $D=4$: the large momentum behavior of propagators associated to internal lines is photon $\sim 1/k^2$, electrons $\sim 1/k$, moreover each vertex gives a $\delta^{(4)}(\sum k_i)$ and at the end a $\delta^{(4)}(\sum \text{external momenta})$ has to remain.

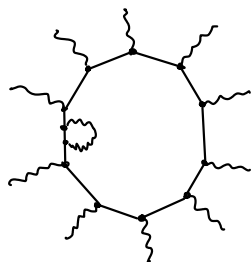
$F_i = \#$ internal fermionic lines ; $B_i = \#$ internal bosonic lines ; $V = \#$ vertices

The superficial degree of divergence of a graph is defined by

$$D_G = 4 \left(\underbrace{F_i + B_i}_{\substack{\# \text{ integrals: } F_i + B_i = \# \text{ currents} \\ V-1 = \# \text{ constraints} \\ \uparrow \text{ global momentum conservation}}} - (V-1) \right) - F_i - 2B_i = 2B_i + 3F_i - 4V + 4$$

and describes the large momentum behavior of the integral: if all momenta are rescaled by S , then the integrand behaves $\sim S^{D_G}$ for $S \rightarrow \infty$. If $D_G > 0$ and no cancellations are present the integral diverges at least as Λ^{D_G} ($\Lambda = UV$ cut-off), if $D_G = 0$ the integral diverges at least as $\log \Lambda$, if $D_G < 0$ possible divergences only come from subdiagrams.

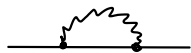
DYSON, BOGOLUBOV, PARASYUK, HEPP, ZIMMERMANN, WEINBERG theorem: if $D_G < 0$ for a diagram and for all its subdiagrams then no divergences are present, moreover if all external (Euclidean) momenta are rescaled by λ , in the $\lambda \rightarrow \infty$ limit the result is λ^{D_G} up to logs (see ITZYKSON, ZUBER "Quantum field theory" § 8.1.4, COLLINS "Renormalization" § 5).



This is a 1PI diagram with superficial degree of divergence

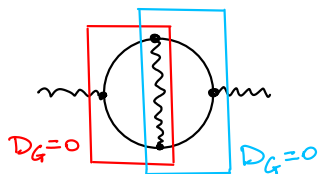
$$D_G = 2B_i + 3F_i - 4V + 4 = 2 + 3 \cdot 2 - 4 \cdot 2 + 4 = -6 < 0$$

but a subdivergence is present in the fermion self-energy



$$D_G = 2 + 3 - 4 \cdot 2 + 4 = +1 > 0$$

In some cases it is not possible to factorize the divergences of subdiagrams (overlapping divergences)



Let us rewrite D_G in a more useful way: the vertex is  and if we draw a dot on each fermionic line close to the vertex  the total number of dots is

$$2V = F_e + 2F_i \quad \text{where } F_e = \# \text{ external fermionic lines}$$

Using photonic lines we find analogously 

$$V = B_e + 2B_i \quad \text{with } B_e = \# \text{ external bosonic lines.}$$

Substituting $B_i = \frac{1}{2}V - \frac{1}{2}B_e$ and $F_i = V - \frac{1}{2}F_e$ in D_G we find

$$D_G = 2B_i + 3F_i - 4V + 4 = V - B_e + 3V - \frac{3}{2}F_e - 4V + 4 = \underline{4 - \frac{3}{2}F_e - B_e}$$

which is independent of the number of vertices and depends only on the number of external legs (note that $\frac{3}{2}$ = mass dimension of the field ψ , 1 = mass dimension of the field A_μ).

The classes of diagrams which can present divergences correspond to $\frac{3}{2}F_e + B_e \leq 4$, and only a finite number of them exists:



(note that gauge invariance induces cancellations

that are not taken into account in the power counting:  $D_G = 0$ but not divergent

and analogously in  only subdivergences are present)

The one loop divergent terms can be represented in the form

$$- \left\{ (Z_A - 1) \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) + (Z_\psi - 1) \bar{\psi} i \not{\partial} \psi - \delta m \bar{\psi} \psi - (Z_e - 1) e \bar{\psi} \not{A} \psi \right\}$$

and by subtracting this expression to $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{\partial} - m) \psi - e \bar{\psi} \not{A} \psi$ we get

$$\mathcal{L}_R = -\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} + Z_\psi \bar{\psi} \left(i \not{\partial} - \frac{m + \delta m}{Z_\psi} \right) \psi - e Z_e \bar{\psi} \not{A} \psi$$

which is the renormalized Lagrangian (with divergent coefficient but finite correlation functions).

If we define $\underline{A_\mu^B = \sqrt{Z_A} A_\mu}$, $\underline{\psi_B = \sqrt{Z_\psi} \psi}$, $\underline{m_B = \frac{m + \delta m}{Z_\psi}}$, $\underline{e_B = e \frac{Z_e}{Z_\psi \sqrt{Z_A}}}$

we get $\mathcal{L}_B = -\frac{1}{4} F_{\mu\nu}^B F_B^{\mu\nu} + \bar{\psi}_B (i \not{\partial} - m_B) \psi_B - e_B \bar{\psi}_B \not{A}_B \psi_B$.

The values of Z_A, Z_ψ can be fixed by imposing asymptotic free field conditions (they are

the factors Z entering the Kallen-Lehman representation and the LSZ reduction formulae), the value of $\bar{m}$ by fixing the physical value of the electron mass and Z_e by fixing the value of the coupling $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$ at the scale m_e . Other less physical procedures of fixing these constant will be discussed in the following. NOTE that Z_e and Z_A are not independent, due to WARD-TAKAHASHI identities.

GENERAL CASE

We consider an interaction of the form $\int \mathcal{L}_I d^D x = g \int F^a B^a d^D x$, where F denotes fermionic fields and B bosonic fields

$d_F = \frac{D-1}{2}$ mass dimension of fermionic fields (from $\int d^D x \bar{\psi} \psi$ dimensionless)

$d_B = \frac{D-2}{2}$ mass dimension of bosonic fields (from $\int d^D x \partial_\mu \phi \partial^\mu \phi$, $\int d^D x F_{\mu\nu} F^{\mu\nu}$ dimensionless)

$d_g = D - f d_F - b d_B =$ mass dimension of the coupling

$$D_G = D(F_i + B_i - (V-1)) - F_i - 2B_i = (D-1)F_i + (D-2)B_i - (V-1)D.$$

Proceeding as in QED we find $fV = F_e + 2F_i$, $bV = B_e + 2B_i$, hence $F_i = \frac{1}{3}fV - \frac{1}{3}F_e$, $B_i = \frac{1}{3}bV - \frac{1}{3}B_e$

$$\text{hence } D_G = (D-1)\frac{1}{3}fV - \frac{D-1}{3}F_e + (D-2)\frac{1}{3}bV - \frac{D-2}{3}B_e - DV + D =$$

$$= -V\left(D - f\frac{D-1}{2} - b\frac{D-2}{2}\right) - \frac{D-1}{2}F_e - \frac{D-2}{2}B_e + D$$

$$D_G = -Vd_g - F_e d_F - B_e d_B + D$$

if $d_g > 0$ only a finite number of divergent 1PI diagrams exists (SUPERRENORMALIZABLE theories)

if $d_g = 0$ on infinite number of divergent 1PI diagrams exists, but only a finite number of classes of divergent diagrams, like in QED (RENORMALIZABLE theories)

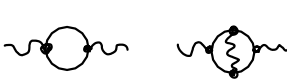
if $d_g < 0$ on infinite number of classes of divergent diagrams exists (NON RENORMALIZABLE theories)

Let us consider some examples

QED $\mathcal{L}_I = -e\bar{\psi}A\psi$ $d_e = D - 2\frac{D-1}{2} - \frac{D-2}{2} = \frac{4-D}{2}$

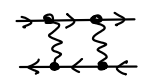
$D < 4$ superrenormalizable: in $D=3$ we have $d_e = \frac{1}{2}$, $d_F = 1$, $d_B = \frac{1}{2}$ and for the photon self-energy ($F_e=0$, $B_e=2$) we have $D_G = -\frac{1}{3}V+2$ hence divergences are present only

for $V=2$ and $V=4$ (V has to be even due to FURRY theorem), corresponding to

diagrams . But $D=2$ we have $d_e = 1$, $d_F = \frac{1}{2}$, $d_B = 0$

$D_G = -V+2$ and only  seems divergent by power counting (it is in fact finite).

$D=4$ renormalizable

$D > 4$ non renormalizable. For example  diverges if $D \geq 6$.

ϕ^4 theory $\mathcal{L}_I = -\frac{g}{4!} \phi^4$ $d_g = D - 4 \frac{D-2}{2} = 4-D$

$D < 4$ superrenormalizable, $D = 4$ renormalizable, $D > 4$ non renormalizable.

ϕ^6 theory $\mathcal{L}_I = -\frac{g}{6!} \phi^6$ $d_g = D - 6 \frac{D-2}{2} = 6-2D$

$D < 3$ superrenormalizable, $D = 3$ renormalizable, $D > 3$ non renormalizable.

Fermi theory $\mathcal{L}_I = -G \bar{\psi}_\mu \psi_\mu$ with $\bar{\psi}_\mu = \bar{\psi} (a \gamma_\mu + b \gamma_\mu \gamma_5) \psi$

$d_G = D - 4 \frac{D-1}{2} = 2-D$

$D < 2$ superrenormalizable

$D = 2$ renormalizable

$D > 2$ non renormalizable

REGULARIZATION

Whatever procedure modifies the UV structure of a theory in such a way to make diagrams convergent is a legitimate regularization method. We work directly in Euclidean space in order to identify more easily the divergent contributions.

Momentum cut-off

We replace $\int_0^\infty \frac{d^D p}{(2\pi)^D}$ by $\int_0^\Lambda \frac{d^D p}{(2\pi)^D}$ (which can be an integral on a sphere or on a cube), where Λ is a UV scale. This regularization breaks the shift-invariance $p \rightarrow p + \delta$ of integrals, gauge invariance (in the gauge transformation there is a derivative, hence momenta change), and possibly rotation invariance (depending on how it is implemented).

Pauli-Villars

We change propagators by using $\frac{1}{p^2 + m^2} \rightarrow \frac{1}{p^2 + m^2} + \sum_i \frac{C_i}{p^2 + m_i^2}$ where C_i and m_i are chosen in such a way as to improve the UV behavior of the propagator, i.e. to make it approach zero faster.

The simplest case is $\frac{1}{p^2 + m^2} \rightarrow \frac{1}{p^2 + m^2} - \frac{1}{p^2 + M^2} = \frac{M^2 - m^2}{(p^2 + m^2)(p^2 + M^2)}$

which makes the propagator behaves like $1/p^4$ for $p^2 \gg M^2$ but does not change the structure for $p^2 \ll M^2$. In the general case more terms are needed, see ITZYKSON, ZUBER "Quantum field theory" § 8.4.2.

Schwinger proper time

We write $\frac{1}{p^2 + m^2} = \int_0^\infty e^{-t(p^2 + m^2)} dt$ then we modify this integral, using a propagator

of the form $\int_0^\infty \rho(t/\Lambda^2) e^{-t(p^2 + m^2)} dt$, with $\rho(x) \rightarrow 0$ for $x \rightarrow 0$ and $\rho(x) \rightarrow 1$ for $x \rightarrow \infty$.

Using for example $\rho(x) = \Theta(x-a)$ we find the propagator $\frac{e^{-(p^2 + m^2)a/\Lambda^2}}{p^2 + m^2}$

Lattice

Instead of regularizing a single diagram we modify the action, by introducing a UV cut-off:

$$S = \int d^D x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi^4 \right\} \rightarrow a^D \sum_{x \in a\mathbb{Z}^D} \left\{ \frac{1}{2} (\nabla_\mu^J \phi_x)^2 + \frac{1}{2} m^2 \phi_x^2 + \frac{g}{4!} \phi_x^4 \right\}$$

with $\nabla_\mu^J \phi_x = \frac{1}{a} (\phi_{x+a\hat{\mu}} - \phi_x)$ and $a = \text{lattice spacing} \sim 1/\Lambda$.

3D Fourier transform $\tilde{\phi}(p) = \frac{a^D}{(2\pi)^D} \sum_x \phi_x e^{-ipx}$ with p in the first Brillouin zone $(-\frac{\pi}{a} \leq p_\mu \leq \frac{\pi}{a})$

The propagator becomes

$$\frac{1}{m^2 + \sum_\mu \frac{4}{a^2} \sin^2\left(\frac{a p_\mu}{2}\right)} \quad \text{which is } \sim \frac{1}{p^2 + m^2} \text{ if } |p_\mu| \ll 1/a$$

PRO: this is a nonperturbative regularization scheme, which allows numerical H.C. simulations and a proper mathematical construction of some superrenormalizable theories.

CON: annoying in perturbative computations (the propagator is a non algebraic function), nontrivial when fermions are present (in particular no known way of lattice regularizing chiral fermions exists), it breaks Lorentz symmetry (but not gauge invariance).

Dimensional regularization

Let us consider for example $I = \int d^D q f(q^2) = \int q^{D-1} f(q^2) dq d\Omega$. If a value of D exist, not necessarily an integer one, which makes the integral finite, we can compute I using that value, and then we analytically continue the result.

Surface of the sphere in D dim:

$$\begin{aligned} \pi^{D/2} &= \left(\int e^{-p^2} dp \right)^D = \int e^{-p^2} d^D p = \Omega(D) \int_0^\infty e^{-p^2} p^{D-1} dp = \frac{\Omega(D)}{2} \int_0^\infty (p^2)^{\frac{D}{2}-1} e^{-p^2} dp^2 = \\ &= \frac{\Omega(D)}{2} \int_0^\infty t^{\frac{D}{2}-1} e^{-t} dt = \frac{1}{2} \Omega(D) \Gamma(D/2), \quad \text{hence } \Omega(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \end{aligned}$$

Let us consider $I = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2}$, which diverges quadratically in $D=4$ but converges, e.g., in $D < 4$.

$$\begin{aligned} I &= \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} = \int \frac{d^D p}{(2\pi)^D} \int_0^\infty dt e^{-t(p^2 + m^2)} \stackrel{(3)}{=} \int_0^\infty dt \frac{e^{-m^2 t}}{(2\pi)^D} \left(\frac{\pi}{t}\right)^{D/2} = \int_0^\infty \frac{dt}{(4\pi)^{D/2}} t^{-D/2} e^{-tm^2} \stackrel{t=y/m^2}{=} \\ &= \frac{1}{(4\pi)^{D/2}} m^{D-2} \int_0^\infty y^{-D/2} e^{-y} dy = \frac{1}{(4\pi)^{D/2}} m^{D-2} \Gamma(1-D/2) \end{aligned}$$

Properties of the Euler Γ function, see appendix for proofs: parts we find $\Gamma(z+1) = z\Gamma(z)$ and if $n \in \mathbb{N}$ $\Gamma(n+1) = n!$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \text{by integrating by}$$

$$\begin{aligned} \Gamma(z) &= \int_0^1 t^{z-1} e^{-t} dt + \int_1^\infty t^{z-1} e^{-t} dt \quad \text{analytic in } z \\ &\quad \uparrow \\ &= \int_0^1 dt \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{t^{k+z-1}}{k!} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_0^1 dt t^{z+k-1} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{1}{z+k} \end{aligned}$$

hence $\Gamma(z)$ is a meromorphic function with simple poles in $z = -k$ ($k \in \mathbb{N}$) and residues $(-1)^k/k!$

From the identity

$$\Gamma(z) = \frac{1}{z} \exp \left\{ -\gamma z + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) z^n \right\} \quad \text{with } \zeta(n) = \sum_{k=1}^{\infty} k^{-n} \text{ Riemann zeta function and}$$

$$\gamma = \lim_{m \rightarrow \infty} \left\{ \sum_{j=1}^m \frac{1}{j} - \log m \right\} \simeq 0,57721... \text{ EULER-MASCHERONI constant}$$

we get

$$\Gamma(z) \simeq \frac{1}{z} - \gamma + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) z + O(z^2)$$

Euler beta function

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$I = \frac{1}{(4\pi)^{D/2}} m^{D-2} \Gamma(1-D/2)$ has poles in $1 - \frac{D}{2} = -n$, hence in $D = 2(1+n)$, hence in $D=2$ and $D=4$ but not in $D=3$. We will soon see that power-like divergences are invisible in dim. reg.

Comparison with momentum cut-off:

$$\underline{D=4}: \frac{1}{(2\pi)^4} \int_0^\Lambda \frac{p^3 dp}{p^2+m^2} \Omega(4) ; \quad \int_0^\Lambda \frac{p^3 dp}{p^2+m^2} = \int_0^\Lambda \frac{p^3}{p^2} dp + \int_0^\Lambda p^3 \left(\frac{1}{p^2+m^2} - \frac{1}{p^2} \right) dp = -\frac{m^2}{\Lambda^2(p^2+m^2)}$$

$$\Omega(4) = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2 \quad = \frac{1}{2} \Lambda^2 - m^2 \int_0^\Lambda \frac{p}{p^2+m^2} dp = \frac{1}{2} \Lambda^2 - \frac{m^2}{2} \log\left(\frac{m^2+\Lambda^2}{m^2}\right)$$

hence $\int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2+m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{m^2}{(4\pi)^2} \log(1+\Lambda^2/m^2) \simeq 2 \log(\Lambda/m)$

$\mathcal{I}_m$ dim. reg $\frac{1}{(4\pi)^{D/2}} m^{D-2} \Gamma(1-D/2) = \frac{1}{(4\pi)^{D/2}} m^{D-2} \frac{1}{1-D/2} \Gamma(2-D/2) \stackrel{D=4}{\simeq} \frac{m^2}{(4\pi)^2} \frac{1}{\frac{D}{2}-2} = -\frac{2m^2}{(4\pi)^2} \frac{1}{4-D}$

$\Gamma(z) = \frac{\Gamma(z+1)}{z}$

$\underline{D=3}: \int_0^\Lambda \frac{p^2 dp}{p^2+m^2} \frac{\Omega(3)}{(2\pi)^3} ; \quad \int_0^\Lambda \frac{p^2 dp}{p^2+m^2} = \int_0^\Lambda \frac{p^2}{p^2} dp + \int_0^\Lambda p^2 \left(\frac{1}{p^2+m^2} - \frac{1}{p^2} \right) dp =$

$$= \Lambda - m^2 \int_0^\Lambda \frac{dp}{p^2+m^2} = \Lambda - m \int_0^{\Lambda/m} \frac{1}{y^2+1} dy = \Lambda - m \arctan\left(\frac{\Lambda}{m}\right)$$

finite

hence in $D=3$ there are no logs.

$\underline{D=2}: \frac{1}{2\pi} \int_0^\Lambda \frac{p dp}{p^2+m^2} = \frac{1}{4\pi} \log(1+\frac{\Lambda^2}{m^2}) \simeq \frac{1}{2\pi} \log \Lambda$ for large Λ

$\mathcal{I}_m$ dim. reg $\frac{1}{(4\pi)^{D/2}} m^{D-2} \Gamma(1-D/2) \stackrel{D=2}{\simeq} \frac{1}{4\pi} \frac{1}{1-D/2} = \frac{1}{2\pi} \frac{1}{2-D}$

Moral: dim. reg. does not see power-like divergences and $\log \Lambda/m \longleftrightarrow \frac{1}{D_c-D}$

To define integrals in D dimensional space (with $D \in \mathbb{R}$ or even $\mathbb{C}$) we use

1) $\int d^D p (a f(p) + b g(p)) = a \int d^D p f(p) + b \int d^D p g(p) \quad 2) \int d^D p f(p+k) = \int d^D p f(p)$

3) $\int d^D p f(\lambda p) = \frac{1}{\lambda^D} \int d^D p f(p) \quad 4) \int d^D p d^N k f(p) g(k) = \int d^D p f(p) \int d^N k g(k)$

4) rotational covariance in D -dim.

and from (2) it follows that $\int \frac{d^D p}{p^{2m}} = 0$ if $2m \neq D$. Note that no real value of D exists which makes this integral finite, since a UV or IR divergence is always present. The result $\int \frac{d^D p}{p^{2m}} = 0$ can be thought as a cancellation between UV and IR divergences:

$$I = \int \frac{d^D p}{p^{2m}} = \Omega(D) \int \frac{p^{D-1} dp}{p^{2m}} = \Omega(D) \left(\int_0^\Lambda p^{D-1-2m} dp + \int_\Lambda^\infty p^{D-1-2m} dp \right)$$

convergent if $D-1-2m > -1$
convergent if $D-1-2m < -1$

and using two different dimensions for the two terms

$$I = \Omega(D) \left(\frac{\Lambda^{D-2m}}{D-2m} - \frac{\Lambda^{D-2m}}{D-2m} \right) \text{ which vanishes when } D_1 = D_2 = D.$$

Hence in dim. reg. UV and IR divergences can mix, something we need to pay attention to.

A consequence of $\int \frac{d^D p}{p^{2m}} = 0$ is that in dim. reg. only log. divergence appear:

$$\int \frac{d^D p}{p^2 + m^2} = \int \frac{d^D p}{p^2 + m^2} - \int \frac{d^D p}{p^2} = -m^2 \int \frac{d^D p}{p^2(m^2 + p^2)} \text{ which is only log. divergent in } D=4.$$

Dim. reg. is thus not only a regularization scheme, but also, in part, a renormalization scheme, since power-law divergences are fixed to zero, and only log. divergences produce poles in $\epsilon=4-D$.

Examples

$$1) I_1 = \int \frac{(p^2)^\beta}{(p^2 + m^2)^\alpha} \frac{d^D p}{(2\pi)^D} = \frac{\Omega(D)}{(2\pi)^D} \int_0^\infty \frac{p^{2\beta+D}}{(p^2 + m^2)^\alpha} \frac{dp}{p} = \frac{\Omega(D)}{(2\pi)^D} m^{2\beta+D-2\alpha} \int_0^\infty \frac{y^{2\beta+D}}{(1+y^2)^\alpha} \frac{dy}{y}$$

Let us now change variable by using $x = \frac{y^2}{1+y^2}$, hence $y^2 = \frac{x}{1-x}$, $1+y^2 = \frac{1}{1-x}$, and $y \frac{dy}{y} = \frac{dx}{x(1-x)}$

$$I_1 = \frac{\Omega(D)}{(2\pi)^D} m^{2\beta+D-2\alpha} \underbrace{\frac{1}{2} \int_0^1 x^{\beta+\frac{D}{2}-1} (1-x)^{\alpha-\beta-\frac{D}{2}-1} dx}_{B(\beta+\frac{D}{2}, \alpha-\beta-\frac{D}{2})} = \frac{m^{2\beta-2\alpha+D}}{(4\pi)^{D/2}} \frac{\Gamma(\beta+\frac{D}{2})\Gamma(\alpha-\beta-\frac{D}{2})}{\Gamma(D/2)\Gamma(\alpha)}$$

$\Omega(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)}$

In particular $\int \frac{1}{(p^2 + m^2)^\alpha} \frac{d^D p}{(2\pi)^D} = \frac{m^{D-2\alpha}}{(4\pi)^{D/2}} \frac{\Gamma(\alpha-D/2)}{\Gamma(\alpha)}$ and we can consider the limit $m \rightarrow 0$:

it is sufficient to use $D > 2\alpha$ (with $\alpha - D/2 \notin \mathbb{N}$, $\alpha > 0$) to see that the limit vanishes, consistently with $\int \frac{d^D p}{p^{2\alpha}} = 0$ found before.

$$2) I_2 = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q+p)^2} = \int dt_1 dt_2 \frac{d^D q}{(2\pi)^D} e^{-t_1 q^2 - t_2 (p+q)^2}$$

$$f(q) = t_1 q^2 + t_2 (p+q)^2; \quad \frac{\partial f}{\partial q} = t_1 q + t_2 (p+q); \quad q_m = -\frac{t_2}{t_1+t_2} p \quad \text{hence } f(q) = (t_1+t_2)(q-q_m)^2 + f(q_m)$$

and $f(q_m) = \frac{t_1 t_2}{(t_1+t_2)} p^2$ hence from the Gaussian integral in q we get

$$I_2 = \frac{\pi^{D/2}}{(2\pi)^D} \int_0^\infty dt_1 dt_2 \frac{1}{(t_1+t_2)^{D/2}} e^{-t_1 t_2 p^2 / (t_1+t_2)}$$

we now use the change of variable $\begin{cases} t_1 = st \\ t_2 = s(1-t) \end{cases}$ with $\begin{cases} s \in (0, \infty) \\ t \in (0, 1) \end{cases}$ $\begin{cases} s = t_1+t_2 \\ t = \frac{t_1}{t_1+t_2} \end{cases}$

$\frac{\partial(t_1, t_2)}{\partial(t, s)} = \begin{pmatrix} s & t \\ -s & 1-t \end{pmatrix} = s$ moreover $t_1 t_2 = s^2 t(1-t)$, hence

$$\begin{aligned}
I_2 &= \frac{1}{(4\pi)^{D/2}} \int_0^1 dt \int_0^\infty ds \, s^{1-D/2} e^{-st(1-t)p^2} \quad \text{and using } s = \frac{u}{t(1-t)p^2} \\
I_2 &= \frac{1}{(4\pi)^{D/2}} (p^2)^{\frac{D}{2}-2} \int_0^1 dt \int_0^\infty du \, u^{1-D/2} e^{-u} [t(1-t)]^{\frac{D}{2}-2} = \\
&= \frac{1}{(4\pi)^{D/2}} (p^2)^{\frac{D}{2}-2} \Gamma(2-\frac{D}{2}) \int_0^1 t^{\frac{D}{2}-2} (1-t)^{\frac{D}{2}-2} dt = \frac{1}{(4\pi)^{D/2}} (p^2)^{\frac{D}{2}-2} \frac{\Gamma(2-D/2)\Gamma^2(D/2-1)}{\Gamma(D-2)} \\
&= B(\frac{D}{2}-1, \frac{D}{2}-1) = \frac{\Gamma^2(\frac{D}{2}-1)}{\Gamma(D-2)}
\end{aligned}$$

with poles in $D=2$ (from IR divergences) and $D=4,6,\dots$ (from UV divergences). To avoid IR div. it is sufficient to add a mass (see the following section), anyway no IR div. are present for $D \geq 4$. $\square$

Let us now discuss in which sense dim. reg. is a regularization, i.e. where is the cut-off scale.

We consider for the sake of the simplicity $I = \int \frac{1}{(p^2+m^2)^\alpha} \frac{d^4 p}{(2\pi)^4}$ and we perform the continuation from 4 to $4-\epsilon$ dimensions: $I_\epsilon = \int \frac{1}{(p^2+m^2)^\alpha} \frac{d^{4-\epsilon} p}{(2\pi)^{4-\epsilon} \mu^{-\epsilon}}$ where μ is an energy scale introduced to keep the physical dimensions constant for any value of ϵ . We can now separate the 4-components from the $(-\epsilon)$ -components, writing

$$I = \int \frac{1}{(p_\epsilon^2 + p^2 + m^2)^\alpha} \frac{d^{4-\epsilon} p_\epsilon}{(2\pi\mu)^\epsilon} \frac{d^4 p}{(2\pi)^4} \quad \text{and using the result of example 1) we have}$$

$$\int \frac{1}{(p_\epsilon^2 + c^2)^\alpha} \frac{d^{4-\epsilon} p_\epsilon}{(2\pi\mu)^\epsilon} = \frac{(c^2)^{-\alpha-\epsilon/2}}{(4\pi\mu^2)^{-\epsilon/2}} \frac{\Gamma(\alpha+\epsilon/2)}{\Gamma(\alpha)} \quad \text{hence } (c^2 = m^2 + p^2)$$

$$I = \frac{\Gamma(\alpha+\epsilon/2)}{\Gamma(\alpha)} \int \frac{1}{(p^2+m^2)^\alpha} \left(\frac{p^2+m^2}{4\pi\mu^2} \right)^{-\epsilon/2} \frac{d^4 p}{(2\pi)^4}. \quad \text{As far as } \alpha > 0 \text{ the ratio } \frac{\Gamma(\alpha+\epsilon/2)}{\Gamma(\alpha)} \text{ approaches 1}$$

when $\epsilon \rightarrow 0$, without generating divergences. The effect of the term

$$\left(\frac{p^2+m^2}{4\pi\mu^2} \right)^{-\epsilon/2} = \exp \left\{ -\frac{\epsilon}{2} \log \left(\frac{p^2+m^2}{4\pi\mu^2} \right) \right\}$$

is to improve the UV behavior for $p^2+m^2 \geq 4\pi\mu^2$ (note: if $m=0$ there are also IR divergences).

$\int$ Integrals with nontrivial tensor structures are extended by covariance, so for example

$$\int p^\mu p^\nu f(p^2) d^4 p = \frac{\delta^{\mu\nu}}{D} \int p^2 f(p^2) d^4 p$$

where we used $\delta^\mu_\mu = D$. We can analogously extend the Clifford algebra of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \mathbb{1}_D \quad (\text{in Euclidean space})$$

from which we get, e.g., $\text{tr}(\gamma^\mu \gamma^\nu) = D\delta^{\mu\nu}$ and $\gamma^\mu \gamma^\nu \gamma_\mu = (2-D)\gamma^\nu$. In dim. reg. some care is instead needed when using γ_5 , since the definition of γ_5 uses $D=4$ in a fundamental way.

A proper extension of the γ_5 properties is required to recover the correct chiral anomaly, see

PESKIN, SCHROEDER "An introduction to quantum field theory" §19.2 and specially COLLINS

"Renormalization" §13, although dim. reg. is not the more transparent way of computing the axial anomaly.

1-LOOP RENORMALIZATION OF ϕ^4 THEORY

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4 \quad \text{Feynman rules: } \text{---}\bullet\text{---} = \frac{1}{p^2+m^2} \quad \text{---}\times\text{---} = -\frac{g}{4!}\delta(\Sigma p_i)$$

We have $d_g = D - 4\frac{D-2}{2} = 4-D$, hence in $D=4$ we get $d_g=0$ and $D_g = 4-B_e$ hence the only primitively divergent graphs are those with $B_e=2$ and $B_o=4$

Let us remember that by writing $\tilde{\Pi}_2 = m^2 + p^2 + \Sigma(p)$ we have $\tilde{W}_2(p) = \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\Sigma(p)\frac{1}{p^2+m^2} + \dots$ hence $\tilde{W}_2(p) = \text{---}\bullet\text{---} + \text{---}\bigcirc\text{---}$ and

$$\tilde{\Gamma}_2(p) = m^2 + p^2 - \bigcirc \quad \text{symmetry factor } 1/2$$

Analogously we have seen in PSF 13 that $\tilde{W}_4 = \text{---}\times\text{---} + \text{---}\bigcirc\text{---} + \text{crossed}$ and

$$\tilde{\Gamma}_4 = g - \bigcirc - \text{crossed}$$

We consider dimensional regularization and the minimal subtraction scheme (MS), in which we subtract only the singular part of the Laurent expansion in $\epsilon = 4-D$ from divergent expression

We later comment of what happens using different regularization/renormalization schemes.

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu \phi_B)^2 + \frac{1}{2}m_B^2\phi_B^2 + \frac{1}{4!}g_B\phi_B^4$$

and we introduce renormalized quantities ϕ, m^2, g by

$$\phi_B = Z_\phi^{1/2}\phi, \quad m_B^2 = m^2 \frac{Z_m}{Z_\phi}, \quad g_B = \mu^\epsilon g \frac{Z_g}{Z_\phi^2} \quad \leftarrow d_{g_B} = 4-D=\epsilon. \text{ In this way } g \text{ is dimensionless in any } D.$$

this is typical of dim. reg. with MS scheme, in which

m^2 renormalizes only multiplicatively. In general there is also an additive term, see R13 for cut-off reg.

By looking at how the different terms enter the computation of the divergence we will also realize that in MS all renormalization constants are independent of m . The MS scheme is thus said to be a mass-independent renormalization scheme.

The renormalization constants Z_ϕ, Z_m, Z_g are fixed by making finite (order by order in the loop expansion) the renormalized correlation functions.

$$W_{2R}(x_1, x_2) = \langle \phi(x_1)\phi(x_2) \rangle_c = \frac{1}{Z_\phi} \langle \phi_B(x_1)\phi_B(x_2) \rangle_c = \frac{1}{Z_\phi} W_{2B}(x_1, x_2)$$

hence $\Gamma_{2R}(x_1, x_2) = Z_\phi \Gamma_{2B}(x_1, x_2)$, and in general we have

$$W_{mB}(x_i, m_B^2, g_B) = Z_\phi^{n/2} W_{mR}(x_i, m^2, g, \mu) \quad \text{and}$$

$$\Gamma_{mR}(x_i, m^2, g, \mu) = Z_\phi^{n/2} \Gamma_{mB}(x_i, m_B^2, g_B) = Z_\phi^{n/2} \Gamma_{mB}(x_i, m^2 \frac{Z_m}{Z_\phi}, \mu^\epsilon g \frac{Z_g}{Z_\phi^2})$$

since Γ_{mR} for $m \gg 2$ is associated to $\perp\!\!\!\perp$ computed correlation functions.

For the 4-dim ϕ^4 theory we thus fix z_ϕ, z_m, z_g by using

$$\tilde{\Gamma}_{2R}(p_i, m^2, g, \mu) = z_\phi \tilde{\Gamma}_{2B}(p_i, m_B^2, g_B)$$

$$\tilde{\Gamma}_{4R}(p_i, m^2, g, \mu) = z_\phi^2 \tilde{\Gamma}_{4B}(p_i, m_B^2, g_B)$$

and the following procedure

- 1) compute (to a given order of the loop expansion) $\tilde{\Gamma}_{2B}$ and $\tilde{\Gamma}_{4B}$ using $\mathcal{L}_B$
- 2) write the previous relations with $m_B^2 = m^2 z_m / z_\phi$ and $g_B = g \mu^\epsilon z_g / z_\phi^2$
- 3) determine z_ϕ, z_m, z_g in such a way that $\tilde{\Gamma}_{2R}$ and $\tilde{\Gamma}_{4R}$ are non-singular.

(a different but equivalent procedure is to write $\mathcal{L}_R = \mathcal{L}_B + \text{counterterms}$, in which $z_{\#-1}$ are the coefficients of the counterterms, add new vertices generated by counterterms, then impose the finiteness of correlation functions. In this way the number of diagrams to be computed increases)

Let us start from the two point function see R.12

$$\tilde{\Gamma}_{2B}(p) = m_B^2 + p^2 + \frac{1}{2} g_B \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m_B^2} = m_B^2 + p^2 + \frac{1}{2} g_B \frac{m_B^{D-2}}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2})$$

$$\boxed{\epsilon = 4 - D}, \quad D = 4 - \epsilon, \quad 1 - \frac{D}{2} = -1 + \frac{\epsilon}{2} \quad \text{and using } \Gamma(1+z) = z\Gamma(z) \text{ we have}$$

$$\Gamma(-1 + \frac{\epsilon}{2}) = \frac{1}{-1 + \epsilon/2} \Gamma(\epsilon/2) \simeq - (1 + \frac{\epsilon}{2}) \left(\frac{2}{\epsilon} - \gamma + O(\epsilon) \right) = - \frac{2}{\epsilon} + \gamma - 1 + O(\epsilon)$$

$$\uparrow \Gamma(z) \simeq \frac{1}{z} - \gamma + O(z) \text{ for } z \approx 0$$

$$m_B^{D-2} = m_B^{2-\epsilon} \quad \text{and using } g_B = \mu^\epsilon g \frac{z_g}{z_\phi^2}, \quad m_B^2 = m^2 \frac{z_m}{z_\phi} \text{ we have}$$

$$m_B^{2-\epsilon} g_B = m_B^2 g \frac{z_g}{z_\phi^2} \left(\frac{\mu}{m_B} \right)^\epsilon = m^2 g \frac{z_m z_g}{z_\phi^3} \left(\frac{\mu}{m} \right)^\epsilon \left(\frac{z_\phi}{z_m} \right)^{\epsilon/2} \quad \text{and } (4\pi)^{D/2} = (4\pi)^{2-\epsilon/2}$$

which together with $\tilde{\Gamma}_{2R} = z_\phi \tilde{\Gamma}_{2B}$ gives

$$\tilde{\Gamma}_{2R} = z_\phi \left(m^2 \frac{z_m}{z_\phi} + p^2 \right) + \frac{z_\phi}{2} m^2 g \frac{(4\pi)^{\epsilon/2}}{(4\pi)^2} \frac{z_m z_g}{z_\phi^3} \left(\frac{\mu}{m} \right)^\epsilon \left(\frac{z_\phi}{z_m} \right)^{\epsilon/2} \left(-\frac{2}{\epsilon} + \gamma - 1 + O(\epsilon) \right)$$

$\uparrow$ only dependence on p^2 , hence $\boxed{z_\phi = 1}$ at this order

$$\tilde{\Gamma}_{2R} = z_m m^2 + p^2 - m^2 g \frac{(4\pi)^{\epsilon/2}}{(4\pi)^2} z_m z_g \left(\frac{\mu}{m} \right)^\epsilon z_m^{-\epsilon/2} \left(\frac{1}{\epsilon} + \frac{1-\gamma}{2} + O(\epsilon) \right)$$

In MS the renormalization constants are of the form $1 + g \frac{C_{11}}{\epsilon} + g^2 \left(\frac{C_{22}}{\epsilon^2} + \frac{C_{21}}{\epsilon} \right) + O(g^3)$

and since in the second term of $\tilde{\Gamma}_{2R}$ a "g" is already present we can use $z_m \simeq z_g \simeq 1$

and from $x^\alpha = e^{\alpha \log x} = (1 + \alpha \log x + O(\alpha^2))$ we get

$$\tilde{\Gamma}_{2R} = z_m m^2 + p^2 - \frac{m^2 g}{(4\pi)^2} \left(1 + \frac{\epsilon}{2} \log(4\pi) + O(\epsilon^2) \right) \left(1 + \epsilon \log \frac{\mu}{m} + O(\epsilon^2) \right) \left(\frac{1}{\epsilon} + \frac{1-\gamma}{2} + O(\epsilon) \right) =$$

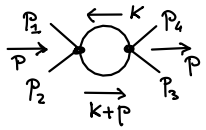
$$= z_m m^2 + p^2 - \frac{m^2 g}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{1}{2} \log(4\pi) + \log \frac{\mu}{m} + \frac{1-\gamma}{2} + O(\epsilon) \right)$$

hence to remove just the $\frac{1}{\epsilon}$ term we must have $\tilde{Z}_m = 1 + \frac{g}{(4\pi)^2} \frac{1}{\epsilon}$ and for $\epsilon=0$ we are left with

$$\begin{aligned}\tilde{\Gamma}_{2R} &= m^2 + p^2 - \frac{m^2 g}{(4\pi)^2} \left(\frac{1}{2} \log(4\pi) + \log \frac{\mu}{m} + \frac{1-\gamma}{2} \right) = \\ &= m^2 + p^2 + \frac{1}{2} \frac{m^2 g}{(4\pi)^2} \left(\log \frac{m^2}{4\pi\mu^2} + \gamma - 1 \right) \quad (\text{MS renorm.})\end{aligned}$$

NOTE: from the previous computation we see that the dependence on μ originates from the development of μ^ϵ , and since g is dimensionless μ enters the divergence only in combinations of the form $\log(m/\mu)$ or $m^2 \log(m/\mu)$. Since dimensional regularization regularizes both UV and IR divergences, if we keep the dimension fixed and consider the limit $m \rightarrow 0$ the divergent term has to be analytic in m , hence the renormalization constants are independent of m and μ (mass independence of MS scheme) and m_B^2 renormalize multiplicatively.

Let us now consider the 4-point functions: from the diagram



, which has symmetry factor $1/2$, and remembering that $\Gamma_m = -\log Z_{m,amp} + \dots$ for $m \geq 2$ we get ($p = p_1 + p_2$)

$$\tilde{\Gamma}_{4B}(p_i, m_B, g_B) = g_B - \frac{1}{2} g_B^2 \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m_B^2} \frac{1}{(k+p)^2 + m_B^2} + p_2 \leftrightarrow p_3 + p_2 \leftrightarrow p_4$$

Let us use the following identity (x, y are called Feynman parameter)

$$\frac{1}{AB} = \frac{1}{B-A} \left(\frac{1}{A} - \frac{1}{B} \right) = \frac{1}{B-A} \frac{1}{xA + (1-x)B} \Big|_{x=0}^1 = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} = \int_0^1 dx dy \delta(x+y-1) \frac{1}{(xA + yB)^2}$$

to rewrite the denominator as

$$\begin{aligned}x[(k+p)^2 + m_B^2] + (1-x)(k^2 + m_B^2) &= \cancel{xk^2} + 2xkp + xp^2 + \cancel{xm_B^2} + k^2 + m_B^2 - \cancel{xk^2} - \cancel{xm_B^2} = \\ &= k^2 + 2xkp + xp^2 + m_B^2 \quad \text{and introducing } l \text{ by } k = l - xp \text{ we have} \\ &= \cancel{l^2 - 2xlp} + \cancel{x^2p^2} + 2xlp - 2x^2p^2 + xp^2 + m_B^2 = l^2 + x(1-x)p^2 + m_B^2\end{aligned}$$

We can thus use example 1) of pag R14 with $d=2$ and $\beta=0$ ($\Gamma(2)=1$):

$$\int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + A^2)^2} = \frac{(A^2)^{(D-4)/2}}{(4\pi)^{D/2}} \Gamma(2 - \frac{D}{2}) \quad \text{where } A^2 = x(1-x)p^2 + m_B^2$$

Using $D=4-\epsilon$ we thus have

$$\begin{aligned}\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m_B^2} \frac{1}{(k+p)^2 + m_B^2} &= \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \int_0^1 (A^2)^{-\epsilon/2} \quad \text{and using } \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon) \text{ we have} \\ \frac{1}{(4\pi)^2} \left(1 + \frac{\epsilon}{2} \log(4\pi) + O(\epsilon^2) \right) \left(\frac{2}{\epsilon} - \gamma + O(\epsilon) \right) \int_0^1 \left(1 - \frac{\epsilon}{2} \log A^2 + O(\epsilon^2) \right) dx &= \\ = \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 \log A^2 dx + O(\epsilon) \right) \quad \text{here we are messing up dimensionalities} \\ &\quad \text{at the end we have to check } \log(\text{dimensionless})\end{aligned}$$

hence

$$\tilde{\Gamma}_{4B} = g_B - \frac{1}{2} \frac{\partial^2 g_B}{(4\pi)^2} \left\{ \frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log [x(1-x)p^2 + m_B^2] + O(\epsilon) \right\} + \text{crossed} =$$

$$= g_B \left\{ 1 - \frac{1}{2} \frac{\partial^2 g_B}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log [x(1-x)p^2 + m_B^2] + O(\epsilon) \right) + \text{crossed} \right\}$$

Remembering that $z_\phi = 1$ at this order and that $g_B = \mu^\epsilon g z_g$ (since $z_\phi = 1$) we have

$$\tilde{\Gamma}_{4R} = \underbrace{\mu^\epsilon g z_g}_{\substack{\uparrow \\ \text{we can use } z_g \approx 1, z_m \approx 1 \text{ since a power of } g \text{ is already present}}}} \left\{ 1 - \frac{1}{2} \frac{\partial^2}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log \left(\frac{x(1-x)p^2 + m^2}{\mu^2} \right) + O(\epsilon) \right) + \text{crossed} \right\}$$

we do not develop this

since it is important to have the correct physical dimensions for $\tilde{\Gamma}_{4B}$

It is immediate to see that in HS we have $z_g = 1 + \frac{3g}{(4\pi)^2} \frac{1}{\epsilon}$ where the 3 comes from the sum on the crossed terms. The physical result is

$$\tilde{\Gamma}_{4R} = g + \frac{3g^2}{2(4\pi)^2} (\gamma - \log(4\pi)) + \frac{g^2}{2(4\pi)^2} \int_0^1 dx \log \left(\frac{x(1-x)p^2 + m^2}{\mu^2} \right) + \text{crossed integrals.} \quad (\text{HS result})$$

NOTE: to compute just z_g we could have simplified the computation by noting that the integral is log divergent, and by performing an expansion in p we see that only the $p=0$ contrib. is divergent, hence we could have fixed $p=0$

To obtain a nontrivial value for z_ϕ we have to go to two loops. Since HS is mass independent we use $m_B = 0$ to simplify the computation. Up to two loops we have

$$\tilde{\Gamma}_{2B} = p^2 - \text{[diagram 1]} - \text{[diagram 2]} - \text{[diagram 3]}$$

and it is immediate to see that in the massless limit the first two diagrams vanishes.

$$\text{[diagram 3]} = \frac{1}{6} g_B^2 \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{k_1^2 k_2^2 (k_1 + k_2 + p)^2} = \frac{1}{6} p^2 \left(-N_D \frac{1}{8\epsilon} + O(1) \right) \quad (\text{see 21NN-SUSTIN § 11.5})$$

where $N_D = \frac{1}{(2\pi)^D} \Omega(D) = \frac{2}{(4\pi)^{D/2} \Gamma(D/2)}$ and $N_4 = \frac{1}{8\pi^2}$. From this we easily get $z_\phi = 1 - \frac{1}{(8\pi^2)^2} \frac{g^2}{48\epsilon}$

In the final expression for $\tilde{\Gamma}_{4R}$ the term with the integral is parametrically larger, for $p^2 \gg \mu^2$, than the tree-level result, thus perturbation theory can fail even if $g \ll 1$.

This can be seen more clearly by explicitly computing the integral:

$$\int_0^1 dx \log \left(\frac{x(1-x)p^2 + m^2}{\mu^2} \right) = \log(m^2/\mu^2) + \int_0^1 dx \log \left(x(1-x) \frac{p^2}{m^2} + 1 \right)$$

moreover we have, with $a = p^2/\mu^2$

$$\int \log(1+x(1-x)a) dx = (x-\frac{1}{2}) \log(1+ax(1-x)) - \frac{\operatorname{atanh}(\sqrt{\frac{a}{a+4}}(1-2x))}{\sqrt{\frac{a}{a+4}}} - 2x$$

thus the definite integral is

$$\int_0^1 \log(1+x(1-x)a) dx = 2 \frac{\operatorname{atanh}(b)}{b} - 2 \quad \text{where we defined } b = \sqrt{\frac{a}{a+4}}.$$

From $\operatorname{atanh} b = c$ we have $b = \operatorname{tanh} c = \frac{e^c - e^{-c}}{e^c + e^{-c}}$, thus $e^c b + e^{-c} b = e^c - e^{-c}$ and $e^c(b-1) + e^{-c}(b+1) = 0$. Thus $e^{2c} = \frac{1+b}{1-b}$ and finally $c = \frac{1}{2} \log \frac{1+b}{1-b}$. Hence

$$\begin{aligned} \log\left(\frac{m^2}{\mu^2}\right) + \int_0^1 \log(1+x(1-x)a) dx &= \log\left(\frac{m^2}{\mu^2}\right) + \log \frac{\sqrt{a+4} + \sqrt{a}}{\sqrt{a+4} - \sqrt{a}} - 2 = \quad \text{remember } a = p^2/m^2 \\ &= \log\left(\frac{m^2}{\mu^2}\right) + \log \frac{\sqrt{4+p^2/m^2} + \sqrt{p^2/m^2}}{\sqrt{4+p^2/m^2} - \sqrt{p^2/m^2}} - 2 \underset{\substack{\uparrow \\ \text{for } p^2 \gg m^2}}{\approx} \log \frac{m^2}{\mu^2} + \log \frac{p^2}{m^2} = \log \frac{p^2}{\mu^2} \end{aligned}$$

which is large if $p^2 \gg \mu^2$.

The region of validity of the expression for $\tilde{T}_{2R}$ is thus $|\log(\frac{m^2+p^2}{\mu^2})| \ll 1$. Parametrically large terms of this sort are known as large-logs, and the perturbative renormalization group was originally developed to cure this problem.

OTHER REG/REN. SCHEMES

A renormalization scheme related to HS is the modified minimal subtraction scheme, denoted by $\overline{\text{MS}}$. In this scheme, beyond the singular terms, we remove also some finite contributions connected to $\frac{1}{\epsilon} \frac{\Omega(D)}{(2\pi)^D}$ and we remember that $\frac{\Omega(D)}{(2\pi)^D} = \frac{2\pi^{D/2}}{\Gamma(D/2)(2\pi)^D} = \frac{2}{(4\pi)^{D/2}} \frac{1}{\Gamma(D/2)}$

using $\boxed{\epsilon = 4-D}$ we have $\frac{D}{2} = 2 - \frac{\epsilon}{2}$ hence

$$(4\pi)^{D/2} = (4\pi)^{2-\epsilon/2} \simeq (4\pi)^2 \left(1 - \frac{\epsilon}{2} \log(4\pi) + O(\epsilon^2)\right) \quad \Gamma(z) = \frac{1}{z} - \gamma + O(z)$$

$$\begin{aligned} \Gamma(D/2) = \Gamma(2 - \frac{\epsilon}{2}) &= \left(1 - \frac{\epsilon}{2}\right) \Gamma(1 - \frac{\epsilon}{2}) = \left(1 - \frac{\epsilon}{2}\right) \left(-\frac{\epsilon}{2}\right) \Gamma(-\frac{\epsilon}{2}) = \left(1 - \frac{\epsilon}{2}\right) \left(-\frac{\epsilon}{2}\right) \left(-\frac{2}{\epsilon} - \gamma + O(\epsilon)\right) = \\ &\quad \Gamma(1+\epsilon) = 2\Gamma(\epsilon) \quad = \left(1 - \frac{\epsilon}{2}\right) \left(1 + \frac{\gamma}{2}\epsilon + O(\epsilon^2)\right) = 1 + \frac{\gamma-1}{2}\epsilon + O(\epsilon^2) \end{aligned}$$

$$\frac{1}{\epsilon} \frac{2}{(4\pi)^{D/2}} \frac{1}{\Gamma(D/2)} = \frac{2}{(4\pi)^2 \epsilon} \left(1 + \frac{\epsilon}{2} \log(4\pi) + O(\epsilon^2)\right) \left(1 - \frac{\epsilon}{2}(\gamma-1) + O(\epsilon^2)\right) =$$

$$= \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} + \underbrace{\log(4\pi) - \gamma + 1}_{\text{in } \overline{\text{MS}} \text{ we remove these terms, which is equivalent to}} + O(\epsilon) \right)$$

redefine $\mu^2 \rightarrow 4\pi\mu^2 e^{-\gamma}$

Cut-off regularization

$$\tilde{T}_{2B}(p) = m_B^2 + p^2 + \frac{1}{2} g_B \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m_B^2} \underset{\text{see pag. R13}}{=} m_B^2 + p^2 + \frac{g_B}{2(4\pi)^2} \left(\Lambda^2 - m_B^2 2 \log \frac{\Lambda}{m_B} \right)$$

$\uparrow$ m_B^2 does not renormalize multiplicatively.

and for $\tilde{T}_{4B}$ we have, after using Feynman parameters and shifting the momenta,

$$\tilde{\Gamma}_{4B}(p) = \partial_B - \frac{1}{2} \partial_B^2 \int_0^1 dx \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 + A^2)^2} + \text{cross} \quad \text{where } A^2 = x(1-x)p^2 + m_B^2$$

$$\Omega(4) = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2 \quad \text{and} \quad \int \frac{\ell^2 d\ell}{(\ell^2 + A^2)^2} = \frac{1}{2} \int \frac{\ell^2 d\ell^2}{(\ell^2 + A^2)^2} \quad \text{moreover}$$

$$\int \frac{y dy}{(y+A^2)^2} = \int \frac{A^2 y - A^2}{(y+A^2)^2} dy = \int \frac{1}{y+A^2} dy - A^2 \int \frac{1}{(y+A^2)^2} dy = \log(A^2 + y) + \frac{A^2}{y+A^2}$$

$$\text{thus } \tilde{\Gamma}_{4B}(p) = \partial_B - \frac{1}{2} \partial_B^2 \frac{1}{(4\pi)^2} \int_0^1 dx \left(\log \frac{\Lambda^2}{x(1-x)p^2 + m_B^2} - 1 \right) + \text{cross}$$

Renormalization constants : we always have

$$\phi_B = Z_\phi^{1/2} \phi$$

$$m_B^2 = \frac{Z_m m^2 + \delta m^2}{Z_\phi}$$

where Z_m renormalizes terms prop to m_B^2 , while δm^2 renormalizes terms prop to Λ^2

while the relation between ∂_B and ∂ can be defined in different ways:

$$\partial_B = \partial \mu^\epsilon Z_\partial / Z_\phi^2$$

introducing the energy scale μ (needed for massless theories)

$$\partial_B = \partial m^\epsilon Z_\partial / Z_\phi^2$$

when we do not want to introduce a new energy scale, so we use

the renormalized mass to fix dimensions (usable only for massive theories)

If we use renormalization conditions consistent with the tree level values of the parameters

we have $Z_\phi, Z_\partial, Z_m = 1 + O(g)$ and $\delta m^2 = O(g)$.

On-shell renormalization scheme : for massive theories we fix the ren. constants in such a way that

$$1) \quad \tilde{\Gamma}_{2R}(p=0) = m^2$$

thus for small p we have $\tilde{\Gamma}_{2R}(p) \simeq m^2 + p^2$ and $\tilde{W}_{2R} \simeq \frac{1}{p^2 + m^2}$

$$2) \quad \left. \frac{\partial}{\partial p^2} \tilde{\Gamma}_{2R}(p) \right|_{p=0} = 1$$

In Minkowski space an analogous requirement would be

to impose $\tilde{W}_{2R} \simeq \frac{i}{p^2 - m^2}$ for $p^2 \simeq m^2$.

$$3) \quad \tilde{\Gamma}_{4R}(p_i=0) = m^4 g$$

In Minkowski space we can use $i \tilde{\Gamma}_{4R}(p_i) = -i m^4 g$, where

momenta p_i are fixed in such a way that $p^2 = m^2$ and $s = 4m^2, t = u = 0$

or using the symmetric point $s = t = u = 4m^2/3$.

This renormalization scheme is "physical" in that all parameters are directly related to some measurable quantities, while in MS the relation is indirect (e.g. m_{MS} is not the pole mass, but obviously it can be connected to it).

NOTE if dim. reg. is used as regulator we have to fix $\mu = m$ in order not to introduce a further energy scale.

This renormalization scheme is quite typical of QED: the pole mass of the electron can be measured (no confinement), and $\alpha \simeq \frac{1}{137}$ is obtained by defining e^2 at vanishing transferred momentum.

In massless theories IR divergences are typically present at zero momentum, so we introduce the energy scale μ and use the renormalization conditions

$$\rightarrow \tilde{T}_{2R}(p=0)=0$$

$$\rightarrow \frac{\partial}{\partial p^2} \tilde{T}_{2R}(p) \Big|_{p^2=\mu^2} = 1$$

$$\rightarrow \tilde{T}_{4R}(p_i|_{sp}) = \mu^\epsilon g$$

where the symmetric point is defined by $\vec{p}_i \cdot \vec{p}_j = \mu^2(\delta_{ij} - \frac{1}{4})$

i.e. $p_i^2 = \frac{3}{4}\mu^2$ and, if $i \neq j$, $(p_i + p_j)^2 = \mu^2$ (all incoming momenta)

Other momenta combinations than sp can be used as far as they are non exceptional (i.e. no sum of a subset of p_i vanishes). The Hinkowski equivalent of this scheme is often called off-shell momentum subtraction scheme (MOH).

Let us consider the massive scheme: using dim. reg. with $\mu=m$ we have (see pag. R14)

$$z_\phi \left(p^2 + m^2 \frac{z_m}{z_\phi} \right) + \frac{z_\phi}{2} m^2 g \frac{(4\pi)^{E/2}}{(4\pi)^2} \frac{z_m z_g}{z_\phi^3} \left(\frac{z_\phi}{z_m} \right)^{E/2} \left(-\frac{2}{\epsilon} + \gamma - 1 + O(\epsilon) \right)$$

hence $z_\phi = 1$ and in the second term a lot of stuff can be put to 1 since a g is already present

$$m^2 z_m \left(1 - \frac{g}{2(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma + 1 + \log(4\pi) \right] \right) + p^2$$

$$\text{hence } z_m = 1 + \frac{g}{2(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma + 1 + \log(4\pi) \right]$$

Using, see pag. R19,

$$\tilde{T}_{4B}(p) = g \left\{ 1 - \frac{1}{2} \frac{g_B}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log [x(1-x)p^2 + m_B^2] + O(\epsilon) \right) + \text{crossed} \right\}$$

and $z_\phi = 1$ we have

$$\tilde{T}_{4R}(p) = m^\epsilon g z_g \left\{ 1 - \frac{1}{2} \frac{g}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log \frac{x(1-x)p^2 + m^2}{m^2} + O(\epsilon) \right) + \text{crossed} \right\}$$

$$\text{hence } z_g = 1 + \frac{3}{2} \frac{g}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) \right) \text{ and } (D=4)$$

$$\tilde{T}_{4R}(p) = g \left\{ 1 + \frac{1}{2} \frac{g}{(4\pi)^2} \int_0^1 dx \log \frac{x(1-x)p^2 + m^2}{m^2} + \text{crossed} \right\}$$

NOTE: large-logs for $p^2 \gg m^2$

Let us now consider a massive theory but renormalized using the condition $\tilde{T}_{4R}(p_i|_{sp}) = \mu^\epsilon g$ instead of $\tilde{T}_{4R}(p_i=0) = m^\epsilon g$: the coupling g is thus defined using the scale μ , while m is independent of it.

Obviously z_ϕ and z_m are the same as before, while for $\tilde{T}_{4R}$ we have

$$\tilde{\Gamma}_{4R}(p) = \mu^\epsilon g z_g \left\{ 1 - \frac{1}{2} \frac{g}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log \frac{x(1-x)p^2 + m^2}{\mu^2} + O(\epsilon) \right) + \text{crossed} \right\}$$

thus (D=4)

$$z_g = 1 + \frac{3}{2} \frac{g}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log \frac{x(1-x)\mu^2 + m^2}{\mu^2} \right)$$

and

$$\tilde{\Gamma}_{4R}(p) = g \left\{ 1 + \frac{1}{2} \frac{g}{(4\pi)^2} \int_0^1 dx \log \frac{x(1-x)p^2 + m^2}{x(1-x)\mu^2 + m^2} + \text{crossed} \right\}$$

NOTE: large logs for $p^2 \gg \mu^2, m^2$
and $\mu^2 \gg p^2, m^2$

RENORMALIZATION GROUP IN MS SCHEME

Let us start by discussing the massless theory, hence

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_B)^2 + \frac{1}{4!} g_B \phi_B^4 = \frac{1}{2} z_\phi (\partial_\mu \phi)^2 + \frac{1}{4!} z_g \mu^\epsilon z_g \phi^4$$

Since $\phi_B = z_\phi^{1/2} \phi$ we have $\tilde{W}_{mB} = z_\phi^{m/2} \tilde{W}_{mR}$ and since $\tilde{\Gamma}_m$ is related to $\tilde{W}_{m,amp}$ we have

$$\tilde{\Gamma}_{mR}(k_i, g, \mu) = z_\phi^{m/2}(g) \tilde{\Gamma}_{mB}(k_i, g_B)$$

and we remember that in MS scheme all ren. constants are mass-independent, so their values are the same in the massive and massless theories. We have

$$0 = \mu \frac{d}{d\mu} \tilde{\Gamma}_{mB} \Big|_{g_B} = \mu \frac{d}{d\mu} (z_\phi^{-m/2}(g) \tilde{\Gamma}_{mR}(k_i, g, \mu)) = z_\phi^{-m/2}(g) \left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{m}{2} \eta(g) \right) \tilde{\Gamma}_{mR}$$

where we introduced $\beta(g) = \mu \frac{\partial g}{\partial \mu} \Big|_{g_B}$ (β -function or, improperly, Gell-Mann, Low β function)

and $\eta(g) = \mu \frac{\partial}{\partial \mu} \log z_\phi(g) \Big|_{g_B} = \beta(g) \frac{\partial}{\partial g} \log z_\phi(g)$ (anomalous dimension of the field ϕ).

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{m}{2} \eta(g) \right) \tilde{\Gamma}_{mR} = 0 \quad \text{is the RG equation in MS scheme}$$

For a perturbative computation of β and η we need the renormalization constants, and we

get β as follows: $0 = \mu \frac{d}{d\mu} g_B \Big|_{g_B} = \mu \frac{d}{d\mu} (g \mu^\epsilon z_g / z_\phi^2) \Big|_{g_B} = \mu^\epsilon \left(\epsilon G(g) + \beta(g) \frac{\partial}{\partial g} G(g) \right)$

where $G(g) = g z_g / z_\phi^2$, hence $\beta(g) = -\epsilon \frac{G(g)}{\partial G / \partial g} = -\epsilon \left(\frac{\partial \log G}{\partial g} \right)^{-1}$.

NOTE β and η are free from divergences: the RG equation for $\tilde{\Gamma}_{4R} / (\tilde{\Gamma}_{2R})^2$ has no anomalous dimension, and since everything is finite also β is finite. Then also η has to be finite.

The finiteness of β and η can be proven by induction, order by order in the loop expansion,

while proving the renormalizability of the ϕ^4 theory, see ZINN-SUSTIN "Quantum field theory and critical phenomena" §10 (scheme different from MS!). For properties of β and η in MS

scheme see ZINN-SUSTIN §11.2-3, WEINBERG "The theory of quantum fields" vol 2 §18.6,

GROSS "Applications of renormalization group in high-energy physics" §4.5-4.6 in BALIAN, ZINN-SUSTIN

"Methods in Field Theory".

We have seen that at one loop $\bar{z}_\phi = 1$ and $\bar{z}_g = 1 + \frac{3}{16\pi^2} \frac{g}{\epsilon}$, at two loop we have

$$\bar{z}_\phi = 1 - \frac{g^2}{48(8\pi^2)^2} \frac{1}{\epsilon}; \quad \bar{z}_g = 1 + \frac{3}{16\pi^2} \frac{g}{\epsilon} + \frac{1}{(8\pi^2)^2} \left(\frac{9}{4\epsilon^2} - \frac{3}{4\epsilon} \right) g^2$$

thus

$$\log G = \log \frac{\bar{z}_g^2}{\bar{z}_\phi^2} = \log \bar{z}_g + \log \bar{z}_g - 2 \log \bar{z}_\phi \quad \text{and using } \log(1+x) \simeq x - \frac{x^2}{2} \text{ we have}$$

$$\log G \simeq \log g + \frac{3}{16\pi^2} \frac{g}{\epsilon} + \frac{1}{(8\pi^2)^2} \left(\frac{9}{4\epsilon^2} - \frac{3}{4\epsilon} \right) g^2 - 2 \frac{g^2}{(16\pi^2)^2} \frac{1}{\epsilon^2} + 2 \frac{g^2}{48(8\pi^2)^2} \frac{1}{\epsilon}$$

$$= \log g + \frac{3}{16\pi^2} \frac{g}{\epsilon} + \frac{1}{(16\pi^2)^2} \frac{g^2}{\epsilon} \left(-3 + \frac{2}{12} \right) + \frac{1}{(16\pi^2)^2} \frac{g^2}{\epsilon^2} \left(9 - \frac{9}{2} \right) \quad \text{hence}$$

$$\left(\frac{2 \log G}{\bar{z}_g} \right)^{-1} = \frac{g}{1 + \frac{3}{16\pi^2} \frac{g}{\epsilon} + \frac{2}{(16\pi^2)^2} \frac{g^2}{\epsilon} \left(-\frac{14}{6} \right) + \frac{2}{(16\pi^2)^2} \frac{g^2}{\epsilon^2} \frac{9}{2}} \simeq \frac{1}{4+x} \simeq 1 - x + x^2$$

$$= g \left(1 - \frac{3}{16\pi^2} \frac{g}{\epsilon} + \frac{17}{3} \frac{1}{(16\pi^2)^2} \frac{g^2}{\epsilon} - \frac{9}{(16\pi^2)^2} \frac{g^2}{\epsilon^2} + \frac{9}{(16\pi^2)^2} \frac{g^2}{\epsilon^2} \right) \quad \text{hence}$$

$$\beta(g) = -\epsilon g + \frac{3}{16\pi^2} g^2 - \frac{17}{3(16\pi^2)^2} g^3$$

tree $\uparrow$ 1 loop $\uparrow$ 2 loop

For the anomalous dimension we have

$$\eta(g) = \beta(g) \frac{2}{g} \log \bar{z}_\phi \simeq (-\epsilon g + O(g^2)) \frac{2}{g} \left(-\frac{g^2}{48(8\pi^2)^2} \frac{1}{\epsilon} \right) = \frac{1}{6(16\pi^2)^2} g^2 \quad \text{2 loop}$$

The RG equation $\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g) \right) \tilde{T}_{mR}(p_i, g, \mu) = 0$ describes how $\tilde{T}_{mR}$ depends on the renormalization scale μ , which is not something of direct physical interest.

To expose the physical content of the RG equation we have to rewrite it using the input of dimensional analysis: $\tilde{T}_{mR}(p_i, g, \mu) = \mu^{d_{\tilde{T}_m}} \hat{T}_{mR}(p_i/\mu, g)$ where $d_{\tilde{T}_m}$ is the mass dimension of $\tilde{T}_m$

To compute $d_{\tilde{T}_m}$ let us remember that the main component of $\tilde{T}_m$ is $\tilde{W}_{m, \text{amp}} = \tilde{W}_m / (\tilde{W}_2)^m$ and $(2\pi)^D \delta(\sum p_i) \tilde{W}_m = \int \prod_i (d^D x_i e^{i p_i x_i}) \langle 0 | T \phi(x_1) \dots \phi(x_m) | 0 \rangle_c$. From $d_\phi = \frac{D-2}{2}$ we get

$$d_{\tilde{W}_m} = m d_\phi - m D + D \quad (\text{the last } +D \text{ is due to } \delta(\sum p_i): [\delta(A)] = [A]^{-1}) \quad \text{and}$$

$$\text{in particular } d_{\tilde{W}_2} = 2 \frac{D-2}{2} - 2D + D = -2 \quad (\text{in free field theory } \tilde{W}_2 = 1/(p^2 + m^2)) \quad \text{hence}$$

$$d_{\tilde{T}_m} = d_{\tilde{W}_m} - m d_{\tilde{W}_2} = m d_\phi - m D + D + 2m = m \frac{D}{2} - m - m D + D + 2m = D \left(1 - \frac{m}{2} \right) + m = 4 - 2m - \epsilon + \epsilon \frac{m}{2} + m = 4 - m + \frac{\epsilon}{2} (m-2) \quad \boxed{D=4-\epsilon}$$

Let us now rescale the momenta by the dimensionless quantity s:

$$\tilde{T}_{mR}(s p_i, g, \mu) = \mu^{d_{\tilde{T}_m}} \hat{T}_{mR}(s p_i/\mu, g)$$

For any function of the form $f(s/\mu)$ we have trivially

$$\left(s \frac{\partial}{\partial s} + \mu \frac{\partial}{\partial \mu}\right) f(s/\mu) = \frac{s}{\mu} f'(s/\mu) - \frac{s}{\mu} f'(s/\mu) = 0$$

thus in particular

$$\left(\mu \frac{\partial}{\partial \mu} + s \frac{\partial}{\partial s}\right) \tilde{\Gamma}_{mR}(sp_i, g, \mu) = \left(\mu \frac{\partial}{\partial \mu} + s \frac{\partial}{\partial s}\right) \left[\mu^{d_{\tilde{\Gamma}_m}} \hat{\Gamma}_{mR}(sp_i/\mu, g) \right] = d_{\tilde{\Gamma}_m} \tilde{\Gamma}_{mR}(sp_i, g, \mu)$$

and we can rewrite

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{m}{2} \eta(g)\right) \tilde{\Gamma}_{mR}(sp_i, g, \mu) = 0$$

in the form

$$\left(-s \frac{\partial}{\partial s} + d_{\tilde{\Gamma}_m} + \beta(g) \frac{\partial}{\partial g} - \frac{m}{2} \eta(g)\right) \tilde{\Gamma}_{mR}(sp_i, g, \mu) = 0 \quad \text{where } d_{\tilde{\Gamma}_m} = 4 - m + \frac{g}{2}(m-2)$$

This equation describes how $\tilde{\Gamma}_{mR}$ changes if we rescale external momenta at fixed renormalization scale μ .

Let us now discuss how to solve this equation: if we use $s = e^{-t}$ (hence $\frac{\partial}{\partial t} = \frac{\partial s}{\partial t} \frac{\partial}{\partial s} = -s \frac{\partial}{\partial s}$)

we have $\left(+\frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} - \frac{m}{2} \eta(g) + d_{\tilde{\Gamma}_m}\right) \tilde{\Gamma}_{mR}(e^{-t} p_i, g, \mu) = 0$ and if we define

$g(t)$ as the solution of the auxiliary ODE $\frac{dg(t)}{dt} = \beta(g(t))$ with initial condition $g(t=0) = g$

we can write the previous equation in the form

$$\left[\frac{d}{dt} - \alpha(g(t))\right] \tilde{\Gamma}_{mR}(e^{-t} p_i, g(t), \mu) = 0 \quad \left(\text{where } \alpha(g(t)) = \frac{m}{2} \eta(g(t)) - d_{\tilde{\Gamma}_m}\right) \quad \text{whose solution is}$$

$$\tilde{\Gamma}_{mR}(e^{-t} p_i, g(t), \mu) = e^{+\int_0^t \alpha(g(t')) dt'} \tilde{\Gamma}_{mR}(p_i, g, \mu) \quad g(t=0) = g$$

If we now rescale momenta by $p_i \rightarrow e^t p_i$ we have

$$\tilde{\Gamma}_{mR}(e^t p_i, g, \mu) = e^{-\int_0^t \alpha(g(t')) dt'} \tilde{\Gamma}_{mR}(p_i, g(t), \mu) = e^{d_{\tilde{\Gamma}_m} t} e^{-\frac{m}{2} \int_0^t \eta(g(t')) dt'} \tilde{\Gamma}_{mR}(p_i, g(t), \mu)$$

and if we use $s = e^t$ (note that this is the inverse of the "s" previously defined, we do not introduce a further symbol not to clutter the notation) we have $dt = ds/s$ and

we can write the solution in the form

$$\tilde{\Gamma}_{mR}(sp_i, g, \mu) = s^{d_{\tilde{\Gamma}_m}} e^{-\frac{m}{2} \int_1^s \eta(g(s')) ds'/s'} \tilde{\Gamma}_{mR}(p_i, g(s), \mu) \quad \text{where } g(s) \text{ is the solution of}$$

$s \frac{d}{ds} g(s) = \beta(g(s))$ with initial condition $g(s=1) = g$. The function $g(s)$ is an effective coupling for the process with momenta sp_i and it is called running coupling constant

NOTE: to explicitly solve the RG equations we used in a fundamental way the fact that β and η depend only on g and not on μ , which is true in MS but not in general schemes.

Depending on the form of the β function several scenarios can happen. Let us consider some examples.

-) Free field theory: $\beta = \eta = 0$, hence $g(s) = g = 0$ and

$$\tilde{T}_{IR}(sp_i, g, \mu) = s^{d_{IR}} \tilde{T}_{IR}(p_i, g, \mu)$$

hence the variation of $\tilde{T}_{IR}$ under a rescaling of momenta is that expected by dimensional analysis.

-) ϕ^4 theory in $D=4$

$$\beta(g) = b_0 g^2 + O(g^3) \quad \text{with } b_0 = \frac{3}{16\pi^2} \quad \text{and} \quad \eta(g) = \gamma_0 g^2 \quad \text{with } \gamma_0 = \frac{1}{6(16\pi^2)^2} g^2$$



$$s \frac{dg(s)}{ds} = b_0 g^2(s) \quad \text{with } g(1) = g; \quad \frac{1}{b_0} \int_g^{g(s)} \frac{dg'}{(g')^2} = \int_1^s \frac{ds'}{s'}; \quad \frac{1}{b_0} \left(\frac{1}{g} - \frac{1}{g(s)} \right) = \log s$$

$$\text{hence } g(s) = \frac{g}{1 - g b_0 \log s} \quad \text{is the running coupling}$$

Since $b_0 > 0$ when increasing s also $g(s)$ increase, and a singularity (the so called Landau pole) seems to exist for finite s , however when $g \approx 1$ the theory becomes nonperturbative and the 1-loop running can not be trusted anymore.

For $s \rightarrow 0$ we have $g \rightarrow 0$ (IR fixed point) and for small s we have

$$g(s) = -\frac{1}{b_0 \log s - 1/g} \simeq -\frac{1}{b_0 \log s} \left(1 + \frac{1}{g b_0 \log s} + \dots \right)$$

and for the integral of the anomalous dimension we have (using only the leading term of $g(s)$)

$$\int_1^s \frac{ds'}{s'} \eta(g(s')) \simeq \frac{\gamma_0}{b_0^2} \int_1^s \frac{ds'}{s' \log^2 s'} = \frac{\gamma_0}{b_0^2} \left(1 - \frac{1}{\log s} \right) \quad \left(\text{using } \int \frac{dx}{x \log^2 x} = -\frac{1}{\log x} \right)$$

Hence, although $g(s) \rightarrow 0$ for $s \rightarrow 0$, there are nontrivial deviations from the free theory, encoded in the logarithmic scaling of the running coupling.

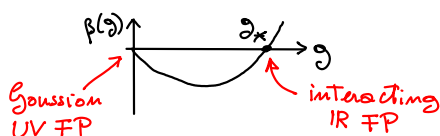
In QED something similar happens, while in Yang-Hills theories and QCD we have $b_0 < 0$ hence we have $g(s) \rightarrow 0$ for $s \rightarrow \infty$ (asymptotic freedom, $g=0$ is a UV Gaussian FP)

$g(s) \rightarrow \text{non pert.}$ for $s \rightarrow 0$ (infrared slavery).

-) ϕ^4 theory in $D=4-\epsilon < 4$

$$\beta(g) = -\epsilon g + b_0 g^2 = b_0 g(g - g_*) \quad \text{with } g_* = \epsilon/b_0 > 0$$

$$\eta(g) = \gamma_0 g^2$$



NOTE: a second FP would be present also in $D=4$ if we consider $O(g^3)$ contribution to $\beta(g)$, but it happens for $g \approx 1$ where pert. theory can not be trusted, while for small ϵ the FP g_* is perturbative

To determine the running coupling we use again $\int_g^{g(s)} dg'/\beta(g') = \int_1^s ds'/s'$ and in this case

$$\frac{1}{\beta(g)} = \frac{1}{b_0 g' (g' - g_*)} = \frac{1}{b_0 g_*} \left(\frac{1}{g' - g_*} - \frac{1}{g'} \right) = \frac{1}{\epsilon} \left(\frac{1}{g' - g_*} - \frac{1}{g'} \right) \quad \text{hence}$$

$$\frac{1}{\epsilon} \left[\log \frac{g(s) - g_*}{g - g_*} - \log \frac{g(s)}{g} \right] = \log s \quad \text{and} \quad \left[\frac{(g(s) - g_*)g}{(g - g_*)g(s)} \right]^{1/\epsilon} = s$$

thus for $s \rightarrow \infty$ we have $g(s) \rightarrow 0$ (asymptotic freedom) and for $s \rightarrow 0$ we have $g \rightarrow g_*$ (nontrivial IR FP: WILSON-FISHER fixed point)

For $s \rightarrow 0$ we have $g \rightarrow g_*$ and consequently $\eta(g(s)) \rightarrow \eta_* \simeq \gamma_0(g_*)^2$. We thus have

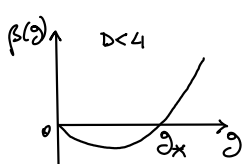
$$\int_1^s \frac{ds'}{s'} \eta(g(s')) \simeq \eta_* \log s$$

hence

$$\tilde{T}_{IR}(sp_i, g, \mu) \simeq S^{d_{IR}} e^{-\frac{\eta_*}{\epsilon} \log s} \tilde{T}_{IR}(p_i, g_*, \mu) = S^{d_{IR} - \frac{\eta_*}{\epsilon}} \tilde{T}_{IR}(p_i, g_*, \mu)$$

and the IR scaling is analogous to that expected from dimensional analysis but for the different value of the exponent, hence the name "anomalous dimension" of $\eta(g)$.

NOTE for $m=2$ $\tilde{T}_{IR}(sp_i, g, \mu) \simeq S^{2-\eta_*}$ hence $\tilde{W}_{IR}(sp_i, g, \mu) \simeq S^{-2+\eta_*}$ and



$$W_{IR}(sp_i, g, \mu) \simeq 1/S^{D-2+\eta_*}$$

In $D < 4$ the physical region $s \in (0, \infty)$ corresponds to $g \in (0, g_*)$ and the $g > g_*$ region can be

reached only through an analytic continuation which introduces singularities. $g > 0$ in $D=4$ corresponds to the limit $D \rightarrow 4$ of $g > g_*$, and this suggests that the ϕ^4 theory in $D=4$ can not be defined beyond perturbation theory.

Let us now consider RG equations in MS scheme for a massive theory: we have

$$\phi_B = Z_\phi^{1/2}(g) \phi, \quad g_B = g \mu^\epsilon Z_g(g) / Z_\phi(g), \quad m_B^2 = m^2 Z_m(g) / Z_\phi(g)$$

and let us introduce for the sake of the simplicity $\tilde{Z}(g) = Z_m(g) / Z_\phi(g)$. Starting

from $\tilde{T}_{IB}(p_i, g_B, m_B^2) = Z_\phi^{-m/2}(g) \tilde{T}_{IR}(p_i, g, m^2, \mu)$ we have (where $|_B$ means that we have to keep constant both g_B and m_B^2)

$$0 = \mu \frac{d}{d\mu} \tilde{T}_{IB} \Big|_B = \mu \frac{d}{d\mu} \left(Z_\phi^{-m/2} \tilde{T}_{IR}(p_i, g, m^2, \mu) \right) = Z_\phi^{-m/2} \left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \left(\mu \frac{d}{d\mu} m^2 \right)_B \frac{\partial}{\partial m^2} - \frac{m}{2} \eta \right] \tilde{T}_{IR}$$

$$\text{moreover } \mu \frac{d}{d\mu} m^2 \Big|_B = \mu \frac{d}{d\mu} (m_B^2 \tilde{Z}^{-1}(g)) \Big|_B = -m_B^2 \frac{1}{\tilde{Z}^2} \mu \frac{d}{d\mu} \tilde{Z} \Big|_B = -m^2 \mu \frac{d}{d\mu} \log \tilde{Z} \Big|_B = -m^2 \eta_2(g)$$

$$\text{where } \eta_2(g) = \mu \frac{d}{d\mu} \log \tilde{Z} \Big|_B = \beta(g) \frac{\partial}{\partial g} \log \tilde{Z}(g) \quad \text{thus}$$

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - m^2 \eta_2(g) \frac{\partial}{\partial m^2} - \frac{m}{2} \eta(g) \right) \tilde{T}_{IR}(p_i, g, m^2, \mu) = 0$$

up to one loop we have $\tilde{z}_\phi = 1$, $\tilde{z}_m = 1 + \frac{g}{(4\pi)^2} \frac{1}{\epsilon}$, thus $\tilde{z} = 1 + \frac{g}{(4\pi)^2} \frac{1}{\epsilon} + O(g^2)$ and

$$\beta(g) = -\epsilon g + \frac{3}{16\pi^2} g^2 + O(g^3) \quad \text{hence}$$

$$\eta_2(g) = \beta(g) \frac{\partial}{\partial g} \log \tilde{z}(g) \simeq \beta(g) \left(\frac{1}{16\pi^2} \frac{1}{\epsilon} + O(g) \right) = -\frac{1}{16\pi^2} g + O(g^2)$$

and using the two loop result one gets

$$\eta_2(g) = -\frac{1}{2(8\pi^2)} g + \frac{5}{24(8\pi^2)^2} g^2$$

To rewrite the RG equation in physical form we can now use

$$\hat{\Gamma}_{mR}(sp_i, g, m^2, \mu) = \mu^{d_{\hat{\Gamma}_m}} \hat{\Gamma}_{mR}(\underbrace{sp_i/\mu}_x, g, \underbrace{m^2/\mu^2}_y)$$

from which we get

$$\begin{aligned} \left(\mu \frac{\partial}{\partial \mu} + s \frac{\partial}{\partial s} + 2m^2 \frac{\partial}{\partial m^2} \right) \hat{\Gamma}_{mR}(sp_i, g, m^2, \mu) &= d_{\hat{\Gamma}_m} \hat{\Gamma}_{mR} + \mu^{d_{\hat{\Gamma}_m}} (\partial_x \hat{\Gamma}_{mR}) \left(-\frac{sp_i}{\mu} \right) + \mu^{d_{\hat{\Gamma}_m}} (\partial_y \hat{\Gamma}_{mR}) (-2m^2/\mu^2) \\ &+ \mu^{d_{\hat{\Gamma}_m}} (\partial_x \hat{\Gamma}_{mR})(sp_i/\mu) + 2\mu^{d_{\hat{\Gamma}_m}} (\partial_y \hat{\Gamma}_{mR})(m^2/\mu^2) = d_{\hat{\Gamma}_m} \hat{\Gamma}_{mR}(sp_i, g, m^2, \mu). \end{aligned}$$

Using this identity we can trade $\mu \frac{\partial}{\partial \mu}$ for $s \frac{\partial}{\partial s}$ and write the RG equation in the form

$$\left[-s \frac{\partial}{\partial s} + d_{\hat{\Gamma}_m} + \beta(g) \frac{\partial}{\partial g} - m^2 (2 + \eta_2(g)) \frac{\partial}{\partial m^2} - \frac{m}{2} \eta(g) \right] \hat{\Gamma}_{mR}(sp_i, g, m^2, \mu) = 0$$

Proceeding as in the massless case we introduce $s = e^{-t}$, hence

$$\left[\frac{\partial}{\partial t} + d_{\hat{\Gamma}_m} + \beta(g) \frac{\partial}{\partial g} - m^2 (2 + \eta_2(g)) \frac{\partial}{\partial m^2} - \frac{m}{2} \eta(g) \right] \hat{\Gamma}_{mR}(e^{-t} p_i, g, m^2, \mu) = 0$$

and by introducing the auxiliary ODEs $\frac{d}{dt} g(t) = \beta(g(t))$ and $\frac{d}{dt} m^2(t) = -(2 + \eta_2(g(t))) m^2(t)$ we have

$$\left[\frac{d}{dt} + d_{\hat{\Gamma}_m} - \frac{m}{2} \eta(g) \right] \hat{\Gamma}_{mR}(e^{-t} p_i, g(t), m^2(t), \mu) = 0$$

and following the same procedure adopted in the massless case we can write the solution of the RG equation in the form

$$\hat{\Gamma}_{mR}(sp_i, g, m^2, \mu) = s^{d_{\hat{\Gamma}_m}} e^{-\frac{m}{2} \int_1^s \eta(g(s')) ds'/s'} \hat{\Gamma}_{mR}(p_i, g(s), m^2(s), \mu)$$

where $g(s)$ and $m^2(s)$ are the solution of the Cauchy problems

$$\begin{cases} s \frac{d}{ds} g(s) = \beta(g(s)), & g(s=1) = g \quad (\text{same eq. as in the massless case}) \\ s \frac{d}{ds} m^2(s) = -m^2(s) (2 + \eta_2(g(s))) & m^2(s=1) = m^2 \quad (\text{running mass}) \end{cases}$$

The second equation can be easily solved:

$$\int_{m^2}^{m^2(s)} \frac{d(m^2)}{(m^2)^2} = - \int_1^s \frac{ds'}{s'} (2 + \eta_2(g(s'))) \quad \text{thus} \quad m^2(s) = m^2 e^{-\int_1^s \frac{ds'}{s'} [2 + \eta_2(g(s'))]}$$

Close to a fixed point $\eta_2(g(s)) \simeq \eta_2(g_*) = \eta_{2*}$ and

$$m^2(s) \simeq m^2 e^{-(2+\eta_{2*}) \log s} = \frac{m^2}{s^{2+\eta_{2*}}} \quad \text{thus if } \eta_{2*} > -2 \text{ we have } m^2(s) \rightarrow 0 \text{ for } s \rightarrow \infty$$

and $m^2(s) \rightarrow \text{large}$ for $s \rightarrow 0$.

Note that the mass running exists even in the free theory, and the "2" comes just from dimensional analysis.

RG EQUATIONS IN OTHER SCHEMES

- 1 Let us consider a scheme in which m is fixed by the renormalization condition (e.g. mass-shell scheme) but the new scale μ is introduced to define g using the symmetric point.

We have at one loop, using dim. reg., see R22-23

$$Z_\phi = 1, \quad Z_m = 1 + \frac{g}{2(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma + 1 + \log(4\pi) \right]$$

$$Z_g = 1 + \frac{3g}{2(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log(x(1-x) + m^2/\mu^2) \right]$$

and we see that this scheme is not mass-independent. The RG equation is

$$0 = \mu \frac{d}{d\mu} \tilde{T}_{MB}(P, g_B, m_B^2) \Big|_{B,m} = \mu \frac{d}{d\mu} \left[Z_\phi^{-m/2}(g, m/\mu) \tilde{T}_{MR}(P_i, g, m^2, \mu) \right] =$$

$$= Z_\phi^{-m/2}(g, m/\mu) \left(\mu \frac{d}{d\mu} + \beta(g, m/\mu) \frac{\partial}{\partial g} - \frac{m}{2} \eta(g, m/\mu) \right) \tilde{T}_{MR}(P_i, g, m^2, \mu)$$

m is independent of μ since it is fixed by a μ -independent ren. condition (using a cut-off m could implicitly depend on Λ , but we are considering derivatives w.r.t. μ , the scale entering the momentum subtraction). This is the GELL-MANN-LAW form of the RG equation.

From the previous equation $\beta(g, m/\mu) = \mu \frac{d}{d\mu} g \Big|_{B,m}$ and

$$\eta(g, m/\mu) = \mu \frac{d}{d\mu} \log Z_\phi(g, m/\mu) \Big|_{B,m} = \beta(g, m/\mu) \frac{\partial}{\partial g} \log Z_\phi + \mu \frac{\partial}{\partial \mu} \log(Z_\phi, m/\mu) \Big|_{B,m}$$

To compute β we can use (introducing for simplicity $G(g, m/\mu) = g Z_g / Z_\phi^2$)

$$0 = \mu \frac{d}{d\mu} g \Big|_{B,m} = \mu \frac{d}{d\mu} (\mu^\epsilon G(g, m/\mu)) \Big|_{B,m} = \mu^\epsilon \left(\epsilon G + \mu \frac{d}{d\mu} G \Big|_{B,m} + \mu \frac{\partial}{\partial \mu} G \Big|_{B,m} \right)$$

$$\text{hence } \beta(g, m/\mu) = \frac{-\epsilon G - \mu \frac{\partial}{\partial \mu} G}{\frac{\partial}{\partial g} G} \quad \text{Using this expression we get, in } D=4 \text{ together with}$$

$$G = g + \frac{3g^2}{2(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 dx \log(x(1-x) + m^2/\mu^2) \right] \quad \text{we find}$$

$$\beta = -\epsilon g \left[1 - \frac{3g^2}{(4\pi)^2} \left(\frac{2}{\epsilon} + O(1) \right) \right] - \epsilon \frac{3g^2}{2(4\pi)^2} \frac{2}{\epsilon} + O(\epsilon) - \frac{3g^2}{2(4\pi)^2} \int_0^1 dx \frac{2m^2/\mu^2}{x(1-x) + m^2/\mu^2} \xrightarrow{\epsilon \rightarrow 0}$$

$$= \frac{3g^2}{(4\pi)^2} \left(1 - \int_0^1 dx \frac{m^2}{x(1-x)\mu^2 + m^2} \right) = \frac{3g^2}{(4\pi)^2} \int_0^1 \frac{x(1-x)\mu^2}{m^2 + x(1-x)\mu^2} dx$$

For $\mu^2 \ll m^2$ we have

$$\beta \simeq \frac{3g^2}{(4\pi)^2} \left(\frac{\mu}{m} \right)^2 \int_0^1 x(1-x) dx = \frac{1}{2(4\pi)^2} g^2 \left(\frac{\mu}{m} \right)^2 \ll g^2$$

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

This behavior is typical of mass-dependent scheme: for $\mu^2 \ll m^2$ there is decoupling

while in mass-independent schemes there is no decoupling.

For $\mu^2 \gg m^2$ we have $\beta = \frac{3g^2}{(4\pi)^2} = \beta_{HS}$, a fact that can be understood as follows:

$\mu^2 \gg m^2$ is equivalent to the $m \rightarrow 0$ limit, hence for dimensional reasons

we have $\beta = b_0 g^2 + O(g^3)$, with $b_0 = \text{pure number}$. Both in MS and in the scheme we are using the renormalized coupling coincides, at tree level, with the parameter entering the Lagrangian, thus $g = g_{\text{MS}} + a_2 g_{\text{MS}}^2 + \dots$. If we have $\beta_{\text{MS}}(g_{\text{MS}}) = b_0^{(\text{MS})} g_{\text{MS}}^2 + O(g_{\text{MS}}^3)$, with $\beta_{\text{MS}}(g_{\text{MS}}) = \mu \frac{d}{d\mu} g_{\text{MS}} \Big|_{\text{B}}$, then

$$\beta(g) = \mu \frac{d}{d\mu} g \Big|_{\text{B}} = \mu \frac{dg}{dg_{\text{MS}}} \Big|_{\text{B}} \frac{dg_{\text{MS}}}{d\mu} \Big|_{\text{B}} = \beta_{\text{MS}}(g_{\text{MS}}) \left(1 + 2a_2 g_{\text{MS}} + O(g_{\text{MS}}^2) \right) =$$

$$= \left(b_0^{(\text{MS})} (g - a_2 g^2) + O(g^3) \right) \left(1 + 2a_2 g + O(g^2) \right) = b_0^{(\text{MS})} g^2 + O(g^3)$$

hence the first coefficient of the massless β function is universal (in fact the first two terms are universal if a single coupling is present, only the first one in theories with multiple couplings).

2. Let us now consider the massive on-shell scheme, in which μ is not present. In this case the RG equation describes the dependence of $\tilde{\Gamma}_{\text{MB}}$ on the physical value of the mass. To get the RG equation we need some preliminaries:

$$\frac{\partial}{\partial m_B^2} \Big|_{g_B} \tilde{\Gamma}_{\text{MB}}(p_1, \dots, p_m) = \tilde{\Gamma}_{(m,1)\text{B}}(0, p_1, \dots, p_m)$$

where $\tilde{\Gamma}_{(m,1)\text{B}}(g, p_1, \dots, p_m)$ is the effective vertex obtained by adding $\frac{1}{2} k(x) \phi^2(x)$ to the Lagrangian density and taking a functional derivative w.r.t. $k(x)$. If we consider the constant limit $k(x) \rightarrow m_B^2$ we get the zero momentum insertion of the operator $+\frac{1}{2} \phi^2(x)$, as can also be understood from the perturbative expression:

$$\frac{\partial}{\partial m_B^2} \frac{1}{p^2 + m_B^2} = \frac{1}{p^2 + m_B^2} (-1) \frac{1}{p^2 + m_B^2} \quad \text{i.e. each propagator (one at time) } \bullet \text{---} \bullet \text{ is replaced by } \bullet \text{---} \boxed{\text{---}} \bullet \text{ with a vertex } -1 \text{ and no ingoing momentum.}$$

The function $\tilde{\Gamma}_{(m,1)\text{B}}(g, p_1, \dots, p_m)$ requires further renormalization w.r.t. $\tilde{\Gamma}_{\text{MB}}$ (we added one parameter, $k(x)$, we need one more renormalization constant), and this is a very particular case of the so called composite operator renormalization. Beyond $\phi_B = Z_\phi^{1/2} \phi$ we also use

$$K_B = k Z_2 / Z_\phi, \quad \text{in such a way that } K_B \phi_B^2 = Z_2 k \phi^2. \quad \text{The generator of renormalized}$$

connected correlation functions is thus $\mathcal{W}_R(\mathcal{J}, K) = \mathcal{W}_B(\mathcal{J}_B, k Z_2 / Z_\phi)$

where $\mathcal{J}_B \phi_B = \mathcal{J}_B Z_\phi^{1/2} \phi \equiv \mathcal{J} \phi$ thus $\mathcal{J}_B = \mathcal{J} / Z_\phi^{1/2}$. To obtain the effective action for the $\frac{1}{2} \phi^2$ insertions we now have to Legendre transform w.r.t. $\mathcal{J}$

$$\Gamma_R[\varphi] = \int \mathcal{S}\varphi - \mathcal{W}_R[\mathcal{S}] = \int \frac{\mathcal{S}}{z_\phi^{1/2}} \varphi z_\phi^{1/2} - \mathcal{W}_B[\mathcal{S}_B] \quad \text{and from } \varphi = \frac{\delta \mathcal{W}_R[\mathcal{S}]}{\delta \mathcal{S}} \text{ we get}$$

$$z_\phi^{1/2} \varphi = \frac{\delta \mathcal{W}_R[\mathcal{S}]}{\delta \mathcal{S}/z_\phi^{1/2}} = \frac{\delta \mathcal{W}_B[\mathcal{S}_B]}{\delta \mathcal{S}_B} = \varphi_B \quad \text{and thus} \quad \Gamma_R[\varphi] = \int \mathcal{S}_B \varphi_B - \mathcal{W}_B[\mathcal{S}_B] = \Gamma_B[\varphi_B]$$

Considering also k we have $\Gamma_R(\varphi, k) = \Gamma_B(\varphi z_\phi^{1/2}, k z_2/z_\phi)$ and thus

$$\tilde{\Gamma}_{(n,1)R} = z_\phi^{m/2-1} z_2 \tilde{\Gamma}_{(n,1)B} \quad \text{which could also have been obtained directly using } \tilde{\Gamma}_{(n,1)} \sim \text{amputated connected functions with insertion.}$$

To fix the value of z_2 we need one further renormalization condition, and since

$$\text{at tree level we have } \tilde{\Gamma}_{2B}(p) = m_B^2 + p^2, \text{ hence } \tilde{\Gamma}_{(2,1)B}(p) = \frac{\partial}{\partial m^2} \tilde{\Gamma}_{2B}(0, p) = 1, \text{ we use the condition } \tilde{\Gamma}_{(n,1)R}(0, p=0) = 1$$

Let us now deduce the RG equations: if we define $D = m \frac{d}{dm} \Big|_{g_B}$ we have

$$D \tilde{\Gamma}_{nB}(p_i, g_B, m_B) = (D m_B^2) \frac{\partial}{\partial m_B^2} \tilde{\Gamma}_{nB}(p_i, g_B, m_B) = (D m_B^2) \tilde{\Gamma}_{(n,1)B}(0, p_i, g_B, m_B)$$

$$\text{moreover } \tilde{\Gamma}_{(n,1)B}(0, p_i, g_B, m_B) = z_\phi^{-m/2+1} z_2^{-1} \tilde{\Gamma}_{(n,1)R}(0, p_i, g, m) \quad \text{and}$$

$$D \tilde{\Gamma}_{nB}(p_i, g_B, m_B) = D \left[z_\phi^{-m/2} \tilde{\Gamma}_{nR}(p_i, g, m) \right] = -\frac{m}{2} \frac{(D z_\phi)}{z_\phi} z_\phi^{-m/2} \tilde{\Gamma}_{nR} + z_\phi^{-m/2} D \tilde{\Gamma}_{nR}(p_i, g, m)$$

$$\text{and } D f(g, m) = m \frac{d}{dm} \Big|_{g_B} f(g, m) = m \frac{\partial}{\partial m} \Big|_g f(g, m) + m \frac{dg}{dm} \Big|_{g_B} \frac{\partial}{\partial g} \Big|_m f(g, m) = \left[m \frac{\partial}{\partial m} \Big|_g + (Dg) \frac{\partial}{\partial g} \Big|_m \right] f(g, m)$$

Thus we have

$$-\frac{m}{2} (D \log z_\phi) z_\phi^{-m/2} \tilde{\Gamma}_{nR} + z_\phi^{-m/2} \left[m \frac{\partial}{\partial m} + (Dg) \frac{\partial}{\partial g} \right] \tilde{\Gamma}_{nR} = z_\phi^{-m/2+1} z_2^{-1} (D m_B^2) \tilde{\Gamma}_{(n,1)R}$$

which by introducing the functions (dimensionless, hence m -independent)

$$\eta(g) = D \log z_\phi; \quad \beta(g) = Dg; \quad m^2 \sigma(g) = z_\phi z_2^{-1} D m_B^2$$

can be written as

$$\left[m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} - \frac{m}{2} \eta(g) \right] \tilde{\Gamma}_{nR}(p_i, g, m) = m^2 \sigma(g) \tilde{\Gamma}_{(n,1)R}(0, p_i, g, m)$$

which is the CALLAN-SYMANZIK equation. For the context in which this equation originates see the lecture "Dilatations" in COLEMAN "Aspects of symmetry", and this equation is typically useful when the r.h.s. can be neglected (typically as a consequence of Weinberg theorem).

3

Let us finally consider the RG equations for the bare quantities, that will be useful when dealing with critical phenomena in statistical systems. We consider a massless theory, regularized with the UV cut-off Λ and renormalized according to the

$$\text{off-shell momentum subtraction scheme: } \tilde{\Gamma}_{2R}(p=0, g, \mu) = 0, \quad \frac{\partial}{\partial p^2} \tilde{\Gamma}_{2R}(p, g, \mu) \Big|_{p^2=\mu^2} = 1$$

$$\tilde{\Gamma}_{4R}(p_i, g, \mu) \Big|_{s_F} = \mu^\epsilon g.$$

$\tilde{\Gamma}_{mR}(p_i, g, \mu, \Lambda) = Z_\phi^{m/2}(g_B, \Lambda/\mu) \tilde{\Gamma}_{mB}(p_i, g_B, \Lambda)$ and $\tilde{\Gamma}_{mR}$ is finite for $\Lambda \rightarrow \infty$ at fixed g, μ .

thus $\Lambda \frac{d}{d\Lambda} \Big|_{g, \mu} \tilde{\Gamma}_{mR} = 0$ for $\Lambda \rightarrow \infty$ (up to corrections which vanish for $\Lambda \rightarrow \infty$, see ZINN-SUSTIN

"Quantum field theory and critical phenomena" § 25.4 for details)

$$\text{Hence } 0 = \Lambda \frac{d}{d\Lambda} \Big|_{g, \mu} (Z_\phi^{m/2} \tilde{\Gamma}_{mB}) = \frac{m}{2} \frac{1}{Z_\phi} \left(\Lambda \frac{d}{d\Lambda} \Big|_{g, \mu} Z_\phi \right) Z_\phi^{m/2} \tilde{\Gamma}_{mB} + Z_\phi^{m/2} \Lambda \frac{\partial}{\partial \Lambda} \Big|_{g, \mu} \tilde{\Gamma}_{mB} + Z_\phi^{m/2} \Lambda \frac{dg_B}{d\Lambda} \Big|_{g, \mu} \frac{\partial}{\partial g_B} \tilde{\Gamma}_{mB}$$

which can be rewritten in the form

$$\left(\Lambda \frac{\partial}{\partial \Lambda} \Big|_{g, \mu} + \beta(g_B, \Lambda/\mu) \frac{\partial}{\partial g_B} - \frac{m}{2} \eta(g_B, \Lambda/\mu) \right) \tilde{\Gamma}_{mB}(p_i, g_B, \Lambda) = 0$$

where

$$\beta(g_B, \Lambda/\mu) = \Lambda \frac{dg_B}{d\Lambda} \Big|_{g, \mu};$$

$$\eta(g_B, \Lambda/\mu) = -\Lambda \frac{d}{d\Lambda} \log Z_\phi(g_B, \Lambda/\mu) \Big|_{g, \mu}$$

4 We have seen that the behavior close to a RG fixed point describes the asymptotic scaling of n -point correlation functions in the limit of large / small momenta. We now want to check that this asymptotic behavior is in fact independent of the specific scheme adopted.

If we consider the UV limit, as seen before, we can set all the masses to zero or use a mass-independent scheme. If we consider the IR limit all massive excitations decouple and we once again remain with only massless excitations; in mass independent schemes this decoupling has to be imposed "by hand". In both the cases we are left with equations of the form $s \frac{d}{ds} g^i(s) = \beta^i(g(s))$, where the index "i" parametrizes the couplings. Let g_*^i be a fixed point of this system, i.e. $\beta^j(g_*) = 0$. Linearizing the evolution equation for $g^i \approx g_*^i$ we get

$$s \frac{d}{ds} (g^i(s) - g_*^i) = \sum_k H^i_k (g^k(s) - g_*^k) \quad \text{where} \quad H^i_k = \left. \frac{\partial \beta^i(g)}{\partial g^k} \right|_{g=g_*}$$

If we denote by $\vec{v}_m$ the eigenvector of H corresponding

to the eigenvalue λ_m , i.e. $\sum_k H^i_k v_m^k = \lambda_m v_m^i$, the solution to the linearized equation can be written in the form $g^i(s) \approx g_*^i + \sum_m v_m^i v_m^j s^{\lambda_m}$ hence the asymptotic behavior for $s \rightarrow 0/\infty$ is determined by the λ_m values.

Let us now see that the eigenvalues λ_m are independent of the choice of the coupling. Let $\tilde{g}^i$ be a new set of couplings, then

$$s \frac{d}{ds} \tilde{g}^i(s) = \sum_m \left. \frac{\partial \tilde{g}^i(g)}{\partial g^m} \right|_{g=g(s)} \beta^m(g(s)) \equiv \tilde{\beta}^i(\tilde{g}(s))$$

hence $\tilde{\beta}^i(\tilde{g}) = \sum_m \frac{\partial \tilde{g}^i}{\partial g^m} \beta^m(g)$ and the β functions transform as contravariant

vectors in the space of couplings. Differentiating w.r.t. g_k we find

$$\sum_m \frac{\partial \tilde{\beta}^i(g)}{\partial \tilde{g}^m} \frac{\partial \tilde{g}^m}{\partial g^k} = \sum_m \frac{\partial^2 \tilde{g}^i}{\partial \tilde{g}^m \partial g^k} \beta^m(g) + \sum_m \frac{\partial \tilde{g}^e}{\partial \tilde{g}^m} \frac{\partial \beta^m(g)}{\partial g^k}$$

At a fixed point $g=g_*$ we have $\beta^m(g_*)=0$ and the previous equations can be written in the form $\tilde{H}S=SH$, where $\tilde{H}$ is the analogous of H for $\tilde{\beta}^i(\tilde{g})$ and

$$S^i_k = \left. \frac{\partial \tilde{g}^i}{\partial g^k} \right|_{g=g_*}, \quad \text{thus as long as } \tilde{g}(g) \text{ is non-singular for } g=g_* \text{ the matrix}$$

S is non-singular and H and $\tilde{H}$ have the same eigenvalues.

GAUSS FUNCTION

(mainly based on PRADISI "lezioni di metodi matematici della fisica" § 3.1 and WHITTAKER, WATSON "A course of modern analysis" § XII)

For $z > 0$ we can define $\Gamma(z) = \int_{-\infty}^{+\infty} t^{z-1} e^{-t} dt$ and by integrating by part it is immediate to see that $\Gamma(1+z) = z\Gamma(z)$, which together with $\Gamma(1)=1$ implies $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt + \int_1^{\infty} t^{z-1} e^{-t} dt \quad \text{analytic for } z \in \mathbb{C}$$

$$\stackrel{z>0}{=} \int_0^1 dt \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{k+z-1}}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{z+k-1} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{z+k}$$

hence $\Gamma(z)$ is a meromorphic function, with simple poles in $z = -k, k \in \mathbb{N}$, and the residue of the pole in $z = -k$ is $(-1)^k/k!$

Let us now see that $\Gamma(z) = \lim_{m \rightarrow \infty} \int_0^m (1-t/m)^m t^{z-1} dt$: we have

$$\int_0^{\infty} e^{-t} t^{z-1} dt - \int_0^m (1-t/m)^m t^{z-1} dt = \int_0^m \{e^{-t} - (1-t/m)^m\} t^{z-1} dt + \int_m^{\infty} e^{-t} t^{z-1} dt.$$

The second integral clearly converges to zero as $m \rightarrow \infty$, and to show that the same is

true also for the first integral we can use the bound $0 \leq e^{-t} - (1-t/m)^m \leq \frac{1}{m} t^2 e^{-t}$

proof of the bound: if $0 \leq y \leq 1$ we have

$e^y = \sum_{n=0}^{\infty} \frac{1}{n!} y^n \geq 1+y$, moreover $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$, hence $1+y \leq e^y \leq \frac{1}{1-y}$. Considering the $-m$ ($m \geq 0$) power of these inequalities we find

$$\left(\frac{1}{1+y}\right)^m \geq e^{-my} \geq (1-y)^m$$

and using $y=t/m$ we get $(1+t/m)^{-m} \geq e^{-t} \geq (1-t/m)^m$ for $0 \leq t \leq m$, hence

in particular $e^t \geq (1+t/m)^m$ and

$$0 \leq e^{-t} - (1-t/m)^m = e^{-t} \{1 - e^t (1-t/m)^m\} \leq e^{-t} \{1 - (1+t/m)^m (1-t/m)^m\} = e^{-t} \{1 - (1-t^2/m^2)^m\}. \quad *$$

if $0 \leq a \leq 1$ we have $(1-a)^m \geq (1-ma)$, indeed this is obviously true for $m=0$ and

$$(1-a)^{m+1} = (1-a)(1-a)^m \geq (1-a)(1-ma) = 1 - (m+1)a + ma^2 \geq 1 - (m+1)a.$$

Using $a = t^2/m^2$ we have

$$1 - (1-t^2/m^2)^m \leq 1 - (1-m \frac{t^2}{m^2}) = t^2/m,$$

and inserting this inequality in $*$ we get $0 \leq e^{-t} - (1-t/m)^m \leq \frac{1}{m} t^2 e^{-t}$ for $0 \leq t \leq m$.

We have $\int_0^m dt (1-t/m)^m t^{z-1} = \int_0^1 m (1-\tau)^m \tau^{z-1} d\tau = m^z \int_0^1 d\tau (1-\tau)^m \tau^{z-1}$ and integrating by part

and using $z > 0, m \in \mathbb{N}$

$$\int_0^1 dz (1-z)^m z^{z-1} = \frac{z^z}{z} (1-z)^m \Big|_0^1 - \int_0^1 dz \frac{z^z}{z} m (1-z)^{m-1} (-1) = \frac{m}{z} \int_0^1 dz (1-z)^{m-1} z^z$$

some as before but for $m \rightarrow m-1$
 $z \rightarrow z+1$

$$= \frac{m(m-1)(m-2) \dots 1}{z(z+1)(z+2) \dots (z+m-1)} \underbrace{\int_0^1 dz z^{z+m-1}}_{1/(z+m)} = \frac{m!}{z(z+1) \dots (z+m)}$$

iterating m times

hence $\Gamma(z) = \lim_{m \rightarrow \infty} \frac{m^z m!}{z(z+1) \dots (z+m)}$. The denominator can be rewritten as

$$z(z+1) \dots (z+m) = z(1+z)z(1+\frac{z}{2}) \dots m(1+\frac{z}{m}) = m! z(1+z)(1+\frac{z}{2}) \dots (1+\frac{z}{m}) \quad \text{hence}$$

$$\Gamma(z) = \lim_{m \rightarrow \infty} \frac{m^z}{z(1+z/2) \dots (1+z/m)} = \frac{e^{z \log m} e^{-z} e^{-z/2} \dots e^{-z/m}}{z \prod_{k=1}^m (1+z/k) e^{-z/k}} =$$

$$= \lim_{m \rightarrow \infty} \frac{\exp\{z [\log m - \sum_{k=1}^m \frac{1}{k}]\}}{z \prod_{k=1}^m (1+z/k) e^{-z/k}} \quad \times^2$$

Let us show that $\sum_{k=1}^m \frac{1}{k} - \log m$ is finite for large m :

$$1 + \frac{1}{2} + \dots + \frac{1}{m} = \int_0^{\infty} (e^{-t} + e^{-2t} + \dots + e^{-mt}) dt = \int_0^{\infty} \sum_{k=1}^m (e^{-t})^k dt = \int_0^{\infty} \frac{e^{-t} - e^{-(m+1)t}}{1 - e^{-t}} dt$$

$$\text{and analogously } \log m = \int_{-1}^m \frac{dx}{x} = \int_{-1}^m dx \int_0^{\infty} dt e^{-tx} = \int_0^{\infty} dt \frac{e^{-t} - e^{-mt}}{t} \quad \text{hence}$$

$$\sum_{k=1}^m \frac{1}{k} - \log m = \int_0^{\infty} dt \left\{ \frac{e^{-t} - e^{-(m+1)t}}{1 - e^{-t}} - \frac{e^{-t} - e^{-mt}}{t} \right\} \text{ is integrable for } t \rightarrow 0, \infty \text{ for any } m, \text{ and in particular}$$

$$\boxed{\gamma} = \lim_{m \rightarrow \infty} \left(\sum_{k=1}^m \frac{1}{k} - \log m \right) = \int_0^{\infty} dt \left\{ \frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-t}}{t} \right\} \approx 0.57721 \quad \text{EULER-MASCHERONI constant.}$$

From eq. $\times^2$ we thus get the Weierstrass representation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z \prod_{k=1}^{\infty} (1 + \frac{z}{k}) e^{-z/k}}$$

The digamma function is defined by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz} \log \Gamma(z)$. Since

$$\log \Gamma(z) = -\log z - \gamma z - \sum_{n=1}^{\infty} \left\{ -\frac{z}{n} + \log(1 + \frac{z}{n}) \right\} \quad \text{we have}$$

$$\psi(z) = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty} \left\{ -\frac{1}{n} + \frac{1}{1+z/n} \frac{1}{n} \right\} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{n+z} \right\}$$

$$\text{Since } \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{k \rightarrow \infty} \sum_{n=1}^k \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{k \rightarrow \infty} \left(\sum_{n=1}^k \frac{1}{n} - \sum_{n=2}^{k+1} \frac{1}{n} \right) = \lim_{k \rightarrow \infty} \left(1 - \frac{1}{k+1} \right) = 1$$

$$\text{we have } \psi(1) = -\gamma.$$

The polygamma function is defined by $\psi^{(k)}(z) = \left(\frac{d}{dz} \right)^k \psi(z) = \left(\frac{d}{dz} \right)^{k+1} \log \Gamma(z)$ with $k > 0$

$$\psi^{(k)}(z) = (-1)^{k+1} \frac{k!}{z^k} - \sum_{n=1}^{\infty} \frac{(-1)^k k!}{(z+n)^k} = (-1)^{k+1} k! \sum_{n=0}^{\infty} \frac{1}{(z+n)^{k+1}}$$

hence

$$\psi^{(k)}(1) = (-1)^{k+1} k! \sum_{n=0}^{\infty} \frac{1}{n^{k+1}} = (-1)^{k+1} k! \zeta(k+1), \text{ where } \zeta(z) \text{ is the Riemann zeta function.}$$

From the values in $z=1$ of the digamma and polygamma and $\Gamma(1)=1$ we get the Taylor series

$$\log \Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) z^n$$

from which

$$\Gamma(z) = \frac{1}{z} \Gamma(z+1) = \frac{1}{z} \left\{ -\gamma z + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) z^n \right\}$$

and for small z we get

$$\Gamma(z) = \frac{1}{z} - \gamma + \frac{1}{2}(\gamma^2 + \zeta(2))z + O(z^2), \text{ where } \zeta(2) = \frac{\pi^2}{6}$$

Since $\int_{-\pi}^{\pi} x \sin(nx) dx = -2\pi \frac{(-1)^n}{n}$ we have the Fourier expansion

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sqrt{\pi}}{n} \frac{\sin(nx)}{\sqrt{\pi}} \quad \text{and using the Parseval identity}$$

$$\frac{2}{3}\pi^3 = \int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \left| \frac{2(-1)^{n+1}\sqrt{\pi}}{n} \right|^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{from which } \zeta(2) = \frac{\pi^2}{6}$$

BETA FUNCTION

The Euler beta function is defined by

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$$

and using $t \rightarrow 1-t$ it is immediate to see that $B(p, q) = B(q, p)$.

Using the change of variable $t = \frac{z}{1+z}$, which maps $(0, \infty)$ to $(0, 1)$, we have

$$1-t = \frac{1}{1+z} \quad \text{and} \quad dt = \frac{1}{(1+z)^2}, \text{ hence}$$

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \int_0^{\infty} \frac{dz}{(1+z)^2} \frac{z^{p-1}}{(1+z)^{q-1}} \frac{1}{(1+z)^{q-1}} = \int_0^{\infty} dz \frac{z^{p-1}}{(1+z)^{p+q}}$$

On the other hand we have

$$\begin{aligned} \Gamma(p)\Gamma(q) &= \int_0^{\infty} dx \int_0^{\infty} dy e^{-x} x^{p-1} e^{-y} y^{q-1} = \int_0^{\infty} dt \int_0^{\infty} dx e^{-x(t+1)} t^{q-1} x^{p+q-1} \\ &= \int_0^{\infty} du \int_0^{\infty} dt e^{-u} t^{q-1} \frac{u^{p+q-1}}{(t+1)^{p+q-1}} \frac{1}{t+1} = \int_0^{\infty} du e^{-u} u^{p+q-1} \int_0^{\infty} dt \frac{t^{q-1}}{(1+t)^{p+q}} \\ &= \Gamma(p+q) B(p, q), \text{ hence } B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \text{ which has simple poles} \end{aligned}$$

if p or q are nonpositive integers.

Laurent expansions of B close to poles can be obtained from that of Γ close to $z=0$.

FEYNMAN PARAMETERS

We want to prove the identity ($\alpha_i > 0, A_i > 0$)

$$\prod_{i=1}^m \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^m \Gamma(\alpha_i)} \int_0^1 \prod_{i=1}^m (dx_i x_i^{\alpha_i-1}) \frac{\delta(1-x)}{(\sum_{i=1}^m x_i A_i)^\alpha} \quad \text{where } x = \sum_{i=1}^m x_i \text{ and } \alpha = \sum_{i=1}^m \alpha_i$$

The integration variables x_i are known as Feynman parameters.

Let us start from the case $n=2$, that will then be needed in the induction step:

$$I_2 = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^1 dx_1 dx_2 \frac{x_1^{\alpha_1-1} x_2^{\alpha_2-1}}{(x_1 A_1 + x_2 A_2)^{\alpha_1 + \alpha_2}} \delta(1-x_1-x_2) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^1 dx_1 \frac{x_1^{\alpha_1-1} (1-x_1)^{\alpha_2-1}}{(x_1 A_1 + (1-x_1) A_2)^{\alpha_1 + \alpha_2}}$$

we use the new variable $z = \frac{x_1 A_1}{x_1 A_1 + (1-x_1) A_2}$, $z \in [0, 1]$, which satisfies

$$1-z = \frac{(1-x_1) A_2}{x_1 A_1 + (1-x_1) A_2}, \quad dz = \frac{A_1 A_2 dx_1}{(x_1 A_1 + (1-x_1) A_2)^2} \quad \text{hence}$$

$$I_2 = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \frac{1}{A_1^{\alpha_1} A_2^{\alpha_2}} \int_0^1 dz z^{\alpha_1-1} (1-z)^{\alpha_2-1} = \frac{1}{A_1^{\alpha_1} A_2^{\alpha_2}} \cdot \underbrace{\int_0^1 dz z^{\alpha_1-1} (1-z)^{\alpha_2-1}}_{B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}}$$

We now assume the identity to hold true for n , and using $n=2$ we show it holds also for $n+1$:

$$\begin{aligned} \frac{1}{A_{m+1}^{\alpha_{m+1}}} \prod_{i=1}^m \frac{1}{A_i^{\alpha_i}} &= \frac{\Gamma(\alpha)}{\prod_{i=1}^m \Gamma(\alpha_i)} \int \prod_{i=1}^m (dx_i x_i^{\alpha_i-1}) \frac{\delta(1-x)}{(\sum_{i=1}^m x_i A_i)^\alpha} \frac{1}{A_{m+1}^{\alpha_{m+1}}} = \\ &= \frac{\Gamma(\alpha)}{\prod_{i=1}^m \Gamma(\alpha_i)} \int_0^1 \prod_{i=1}^m (dx_i x_i^{\alpha_i-1}) \delta(1-x) \frac{\Gamma(\alpha + \alpha_{m+1})}{\Gamma(\alpha) \Gamma(\alpha_{m+1})} \int_0^1 dx_{m+1} dy \frac{y^{\alpha-1} x_{m+1}^{\alpha_{m+1}-1} \delta(1-y-x_{m+1})}{(y \sum_{i=1}^m x_i A_i + x_{m+1} A_{m+1})^{\alpha + \alpha_{m+1}}} \end{aligned}$$

If we use $x_i = x'_i / y$ for $i=1, \dots, m$ we have $\delta(1-x) = \delta(1-x'/y) = y \delta(y-x')$ and

moreover $\prod_{i=1}^m dx_i x_i^{\alpha_i-1} = \frac{1}{y^\alpha} \prod_{i=1}^m dx'_i (x'_i)^{\alpha_i-1}$ and by integrating on y we get

$$\frac{\Gamma(\alpha + \alpha_{m+1})}{\prod_{i=1}^{m+1} \Gamma(\alpha_i)} \int \prod_{i=1}^m (dx'_i (x'_i)^{\alpha_i-1}) \frac{dx_{m+1} x_{m+1}^{\alpha_{m+1}-1} \delta(1-x'-x_{m+1})}{(\sum_{i=1}^m x'_i A_i + x_{m+1} A_{m+1})^{\alpha + \alpha_{m+1}}}$$

which is the induction step after labelling $x'_i \rightarrow x_i$ for $i=1, \dots, m$.

Lie groups & representations

Group: set G with a map $G \times G \rightarrow G$ which satisfies

- 1) $g_1(g_2g_3) = (g_1g_2)g_3$ for any $g_1, g_2, g_3 \in G$ (associativity)
- 2) $e \in G$ exists such that $ge = eg = g$ for any $g \in G$ (existence of the identity element)
- 3) for any $g \in G$ an element g^{-1} exists such that $gg^{-1} = g^{-1}g = e$ (existence of the inverse).

If $g_1g_2 = g_2g_1$ for any $g_1, g_2 \in G$ then G is said to be an Abelian group.

Lie group: a group that is also a C^∞ manifold, in which elementary operations (products and inverse) are C^∞ maps. The dimension of the group is the dimension of the manifold, i.e. the number of parameters that are needed to univocally identify a group element.

The group is said to be compact if the manifold is compact.

EXAMPLES

- 1) $GL(n, \mathbb{C})$: group of invertible linear transformations in $\mathbb{C}^n$, i.e. group of $n \times n$ invertible matrices with complex entries. The defining equation is $\det H \neq 0$, and $\det H = 0$ represent a codimension 1 hypersurface in $\mathbb{C}^n$, hence $GL(n, \mathbb{C})$ is an open subset of $\mathbb{C}^n$ parametrized by $2n^2$ numbers.
- 2) $O(n)$ orthogonal group: group of isometries of $\mathbb{R}^n$ with scalar product $\langle x, y \rangle = \sum x_i y_i$, i.e. group of $n \times n$ real matrices with $O^t O = 1$ (note: $\det O = \pm 1$)
Since $(O^t O)^t = O^t O$ the constraint $O^t O = 1$ is equivalent to $n + \frac{n^2-1}{2}$ equations, and the dimensionality of $O(n)$ is thus $n^2 - (n + \frac{n^2-1}{2}) = \frac{n(n-1)}{2}$ and $O(n)$ is a compact group
- 3) $SO(n)$ special orthogonal group: group of proper isometries of $\mathbb{R}^n$, i.e. $n \times n$ real matrices satisfying $O^t O = 1$ and $\det O = 1$. Since $\det O = \pm 1$ in $O(n)$, we have $\dim O(n) = \dim SO(n)$ and also $SO(n)$ is a compact group.
- 4) $U(n)$ unitary group: group of isometries of $\mathbb{C}^n$ with scalar product $\langle x, y \rangle = \sum x_i^* y_i$, i.e. group of $n \times n$ complex matrices such that $U^\dagger U = 1$ (note $|\det U| = 1$)
Since $(U^\dagger U)^\dagger = U^\dagger U$ the constraint $U^\dagger U = 1$ is equivalent to $n + 2 \frac{n^2-n}{2}$ real equations, and $\dim U(n) = 2n^2 - n^2 = n^2$. Also $U(n)$ is a compact group.
- 5) $SU(n)$ special unitary group: $n \times n$ complex matrices with $U^\dagger U = 1$ and $\det U = 1$. From the further constraint $\det U = 1$ we get $\dim SU(n) = \dim U(n) - 1 = n^2 - 1$ and also $SU(n)$ is compact.

Group elements close to the identity can be parametrized as

$$g = e - \sum_a T^a d\theta^a \quad \text{with } a=1, \dots, \dim(G)$$

small parameters which identify the group element

generators of the group (basis of the tangent space at G in e)

Since if g and g_1 are close to the identity also $g_1^{-1}g$ is close to the identity we have that the generators satisfy the commutation rules

$$[T^a, T^b] = f^{abc} T^c \quad \text{Lie algebra of the group}$$

where f^{abc} are the structure constants of the group, which uniquely identify the Lie algebra and almost the group. Note that the definitions of the generators and of the structure constants have some degrees of freedom: for complex groups the definitions $g = e - \sum_a i T^a d\theta^a$ and $[T^a, T^b] = i f^{abc} T^c$ are typically used, and the numerical values of the structure constants depend on the normalization adopted for the generators.

EXAMPLES

→ $U(1)$: $g = e^{-i\theta} \simeq 1 - i\theta$ and $T^1 = 1$

→ $SO(2)$: $g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \simeq 1 - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \theta$ hence $T^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T^1_{ij} = \epsilon_{ij}$

→ $SO(3)$: $SO(3)$ is generated by rotation along the axes (Euler angles parametrization) and an infinitesimal righthanded rotation of angle θ around $\hat{n}$ is given by

$$\vec{x} \rightarrow \vec{x} - \theta \hat{n} \times \vec{x} \quad \text{hence } \delta x_i = -\theta \epsilon_{ijk} n_j x_k = -\theta n_j (\epsilon_{ijk}) x_k \quad \text{hence we can use}$$

$$(T^a)_{ij} = \epsilon_{iaj}, \quad \text{from which}$$

$$\begin{aligned} (T^a T^b - T^b T^a)_{ij} &= \epsilon_{iak} \epsilon_{kbj} - \epsilon_{ibk} \epsilon_{kaj} = \delta_{ib} \delta_{aj} - \delta_{ij} \delta_{ab} - \delta_{ia} \delta_{bj} + \delta_{ij} \delta_{ab} = \\ &= \epsilon_{cab} \epsilon_{cji} = \epsilon_{abc} (T^c)_{ij} \end{aligned}$$

hence using these generators the structure constants are $f_{abc} = \epsilon_{abc}$

→ $SU(2)$: if we write $g = e^{-iT^a \theta^a} \simeq 1 - iT^a \theta^a$ from $g^\dagger g = 1$ we find $(T^a)^\dagger = T^a$, moreover $\det e^H = e^{\lambda_1 + \lambda_2 + \dots + \lambda_n} = e^{\text{tr} H}$, hence $\det g = 1$ implies $\text{tr}(T^a) = 0$.

Pauli matrices constitutes a basis for Hermitian traceless 2×2 matrices and

$$\text{we can use } T^a = \frac{1}{2} \sigma^a \quad (\text{the } \frac{1}{2} \text{ is conventional}), \quad \text{hence } [T^a, T^b] = i \epsilon^{abc} T^c$$

and the structure constants of $SU(2)$ and $SO(3)$ are the same.

NOTE: the fact that two groups share the same structure constants means that the two

groups are locally (i.e. close to the identity) the same. The two groups can differ only

by global properties: $SO(3)$ is not simply connected ($\pi_1(SO(3)) = \mathbb{Z}_2$) while $SU(2)$

is simply connected (it is the double covering of $SO(3)$).

-) $SU(n)$ proceeding in the same way as done for $SU(2)$ we see that T^a has to be a basis for Hermitian traceless matrices. Note that the dimensionality of the space of Hermitian traceless matrices is

$$\underbrace{2}_{\text{complex numbers}} \underbrace{\frac{n^2-n}{2}}_{\text{upper triang. part}} + \underbrace{n-1}_{\text{diagonal}} = n^2-1 \quad \text{consistently with the previous computation of } \dim SU(n)$$

n real numbers with the constraint $\text{tr}=0$

Generators of $SU(N)$ are typically normalized in such a way that $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$

-) $SO(n)$: writing $g = e^{\vec{T}^a \theta^a} \approx 1 - \vec{T}^a \theta^a$ from $g^t g = 1$ we find $(T^a)^t = -T^a$, which implies that $\text{tr}(T^a) = 0$ and thus $\det g = 1$. For an antisymmetric matrix we have $\text{tr}(A^2) = -\text{tr}(A^t A) = -(A^t)_{ij} A_{ji} = -\sum_{ij} (A_{ji})^2 < 0$

so the generators are typically normalized by using $\text{tr}(T^a T^b) = -2 \delta^{ab}$.

From the Jacobi identity $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$ the Jacobi identity for the structure constants easily follows

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0$$

For compact semisimple groups (compact groups whose algebra does not have Abelian invariant sub-algebras) it is possible to define the generators in such a way that they satisfy $\text{tr}(T^a T^b) = K \delta^{ab}$, hence

$$f^{abc} = \frac{1}{K} \text{tr}([T^a, T^b] T^c)$$

see e.g. $SU(n)$ and $SO(n)$ above

and it is immediate to show that the structure constants are completely antisymmetric, e.g.

$$f^{abc} = \frac{1}{K} \text{tr}(T^a T^b T^c - T^b T^a T^c) = \frac{1}{K} \text{tr}(T^b T^c T^a - T^c T^b T^a) = -\frac{1}{K} \text{tr}([T^c, T^b] T^a) = -f^{cba}$$

For compact groups a unique left and right invariant measure exists, which satisfies

$$dg = d(g_1 g) = d(g g_1) \quad \text{where } g_1 \text{ is a fixed element of } G$$

This measure is called the HAAR measure and it is normalized in such a way that $\int_G dg = 1$.

A representation of rank N of the group G is a C^∞ map from G to $GL(N, \mathbb{C})$ (or $GL(N, \mathbb{R})$ in real case) which satisfies

$$\cdot) \quad R(e) = \mathbb{1}_{N \times N}$$

$$\cdot) \quad R(g_1) R(g_2) = R(g_1 g_2) \quad \text{for } g_1, g_2 \in G \quad (\text{in particular } R(g^{-1}) = R(g)^{-1})$$

and it is thus an isomorphism between G and a subset of $GL(N, \mathbb{C})$.

A representation is called reducible if a nontrivial linear subspace of $\mathbb{C}^N$ is left invariant by the action of the group (i.e. $\forall \varphi \in \mathbb{C}^N$ exists such that $R(g)\varphi \in V$ for any $\varphi \in V$ and $g \in G$).

A reducible representation can be written in block form as $R(g) = \begin{pmatrix} X(g) & Y(g) \\ 0 & Z(g) \end{pmatrix}$ and representations that are not reducible are called irreducible.

Unitary or orthogonal representations are those for which $R(g) \in U(N)$ or $R(g) \in O(N)$.

In these cases it is simple to verify that if $V \subset \mathbb{C}^N$ or $\mathbb{R}^N$ is an invariant subspace, then also $V^\perp$ is invariant, and unitary/orthogonal reducible representations are thus completely reducible, i.e. they can be written in block diagonal form.

An useful fact about irreducible representation is SCHUR lemma: if $R(g) \in GL(N, \mathbb{C})$ is an irreducible representation and M is a $N \times N$ matrix, then $[R(g), M] = 0$ for all $g \in G$ implies that $M = \lambda \mathbb{1}$ for $\lambda \in \mathbb{C}$

dim: if $\vec{v}$ is an eigenvector of M , $M\vec{v} = \lambda\vec{v}$, then $M R(g)\vec{v} = R(g) M\vec{v} = \lambda R(g)\vec{v}$, hence the eigenspace V_λ corresponding to λ is an invariant subspace for the irr. rep, hence or $V_\lambda = \{0\}$ or $V_\lambda = \mathbb{C}^N$. Since $V_\lambda \neq \{0\}$ by hypothesis we have $M = \lambda \mathbb{1}$ $\square$

As a consequence irreducible representations of Abelian groups are one-dimensional.

For g close to the identity we have

$$R(g) \simeq \mathbb{1}_{N \times N} - T_a^{(R)} \theta^a \quad (\text{or } \mathbb{1}_{N \times N} - i T_a^{(R)} \theta^a \text{ in the complex case})$$

where $T_a^{(R)}$ are the generators of the representation R .

The fundamental, or defining, representation is the one used to define the group: e.g. for $SU(N)$ the defining representation acts on $\mathbb{C}^N$, and the generators are $N \times N$ Hermitian traceless matrices. Representations of higher rank exists, in which $T_a^{(R)}$ are $N \times N$ Hermitian traceless matrices, with $N > N$.

$SU(2)$: representations of $SU(2)$ are parametrized by the spin. Integer spin representations of $SU(2)$ are also representations of $SO(3)$

$$\text{spin } 1/2: \quad \vec{\psi} \in \mathbb{C}^2, \quad \vec{\psi} \rightarrow e^{-i\vec{\theta} \cdot \vec{\sigma}/2} \vec{\psi}$$

$$\text{spin } 1: \quad \vec{\psi} \in \mathbb{R}^3 \quad (\text{since the transformation matrix is real}), \quad \vec{\psi} \rightarrow e^{-i\vec{\theta} \cdot \vec{T}^a} \vec{\psi}$$

$$\text{with } (T^a)_{bc} = i \epsilon^{bac} \quad \text{structure constants of } SU(2)$$

$$= \text{fundamental representation of } SO(3).$$

If T^a are the $n \times n$ generators of $SU(N)$, $R(g) = e^{-i\vec{\theta} \cdot \vec{T}^a}$ is a $SU(N)$ matrix and let us consider $M = R^\dagger(g) T^a R(g)$: clearly $M^\dagger = M$ and $\text{tr} M = \text{tr} T^a = 0$, hence we can expand M on the basis $\{T^b\}_{b=1}^{N^2-1}$ and write $R^\dagger(g) T^a R(g) = M_{ab}(g) T^b$, where M_{ab} is a $(N^2-1) \times (N^2-1)$ matrix.

We also have $R^\dagger(\vartheta_1 \vartheta_2) T_a R(\vartheta_1 \vartheta_2) = [R(\vartheta_1) R(\vartheta_2)]^\dagger T_a R(\vartheta_1) R(\vartheta_2) = R^\dagger(\vartheta_2) R^\dagger(\vartheta_1) T_a R(\vartheta_1) R(\vartheta_2) =$
 $= R^\dagger(\vartheta_2) M_{ab}(\vartheta_1) T_b R(\vartheta_1) = M_{ab}(\vartheta_1) M_{bc}(\vartheta_2) T^c$, moreover $R^\dagger(\vartheta_1 \vartheta_2) T_a R(\vartheta_1 \vartheta_2) = M_{ac}(\vartheta_1 \vartheta_2) T^c$,
hence $M(\vartheta_1 \vartheta_2) = M(\vartheta_1) M(\vartheta_2)$, and obviously $M(e) = \mathbb{1}$. Hence $M(g)$ is a representation
of the group $SU(N)$, which is called the adjoint representation. To find the generators of
the adjoint representation we can use the infinitesimal form of $R(g)$:

$$R^\dagger(g) T^a R(g) \simeq (1 + i T^b \theta^b) T^a (1 - i T^c \theta^c) \simeq T^a + i [T^c, T^a] \theta^c = T^a + i f^{cab} \theta^c T^b =$$

$$= T^a - i (i f^{acb}) \theta^c T^b$$

hence $(T_{adj}^c)_{ab} = i f^{acb}$ (compare with spin 1 repr. of $SU(2)$)

Global symmetries and spontaneous symmetry breaking

We consider global continuous internal symmetries (internal = independent of space-time geometry). For example the N -component real scalar ϕ^4 theory with Euclidean Lagrangian density

$$\mathcal{L} = \frac{1}{2} \sum_{\mu} (\partial_{\mu} \vec{\phi}) \cdot (\partial_{\mu} \vec{\phi}) + \frac{1}{2} \pi |\vec{\phi}|^2 + \frac{\lambda}{4!} |\vec{\phi}|^4 = \frac{1}{2} \sum_{\mu} \sum_i (\partial_{\mu} \phi_i)^2 + \frac{1}{2} \pi \sum_i \phi_i^2 + \frac{\lambda}{4!} \left(\sum_i \phi_i^2 \right)^2$$

is invariant under the $O(N)$ transformation $\vec{\phi}(x) \rightarrow O \vec{\phi}(x)$, with the $N \times N$ matrix O being independent of x . The infinitesimal form of an internal continuous global symmetry can be written

$$\delta \phi_i(x) = -T_{ij}^a \phi_j(x) \varepsilon^a$$

T^a = generators (in the appropriate rep.) of the symmetry

ε^a = parameters of the transformation

If $\mathcal{L}$ is invariant under the field variation $\vec{\phi}(x) \rightarrow \vec{\phi}(x) + \delta \vec{\phi}$ we have in classical field theory the Noether conserved current

$$J^{\mu a} = - \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_i)} \frac{\delta \phi_i}{\delta \varepsilon^a} :$$

$$-\partial_{\mu} J^{\mu a} = \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_i)} \right) \frac{\delta \phi_i}{\delta \varepsilon^a} + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_i)} \partial_{\mu} \frac{\delta \phi_i}{\delta \varepsilon^a} = \frac{\partial \mathcal{L}}{\partial \phi_i} \frac{\delta \phi_i}{\delta \varepsilon^a} + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_i)} \frac{\delta \partial_{\mu} \phi_i}{\delta \varepsilon^a} = \frac{\delta}{\delta \varepsilon^a} \mathcal{L} = 0$$

and the associated conserved charges $Q^a = \int d^3x J_0^a(x, t)$ (in which we assume currents to vanish sufficiently fast at infinity for the integral to be convergent)

$$\frac{d}{dt} Q^a = \int d^3x \partial_0 J_0^a = \int d^3x (\partial^{\mu} J_{\mu}^a - \partial^i J_i^a) = - \int_{\Sigma} \vec{J}^a \cdot d\vec{\Sigma} = 0 \text{ if there are no currents at infinity.}$$

What happens in quantum field theory? There are two possible problems

1) Does a quantum counterpart of the conserved Noether current exist?

In some cases symmetries do not survive quantization (anomalies). This is typically related to the peculiarities of the regularization/renormalization procedure: e.g. (flavor singlet)

chiral symmetry and the invariance under rescaling require no mass scales to be present in the theory, but in the reg/ren. process mass scales are introduced. In these cases we have $\partial_\mu S^\mu = \alpha(h) \neq 0$ and the symmetry is explicitly broken in the quantum theory.

2) When $\partial_\mu S^\mu$ holds as a local equation, is the associated conserved charge well defined and the symmetry associated to a unitary operator in the Hilbert space?

Both the cases are possible: if the answer is positive the symmetry is said to be realized à la WIGNER-WEYL (or sometimes is said to be realized in the spectrum), otherwise the symmetry is said to be spontaneously broken.

Let us start considering the Wigner-Weyl case: the charge operators are well defined and annihilate the vacuum $Q^a |0\rangle = 0$ (we will see later that if this does not happen we have spontaneous symmetry breaking). The charge operators generate the unitary transformations $U(\vec{E}) = \exp(-i \vec{E}^a Q^a)$, which commute with the Hamiltonian and leave the vacuum invariant. Charge operators can be written in the form $Q^a = \int d^3x \Pi^i(x) T^a_{ij} \phi^j(x)$, where Π^i are the conjugate momenta of ϕ^i , and we have (at fixed time)

$$[Q^a, \phi^k(y)] = \int d^3x [\Pi^i(x) T^a_{ij} \phi^j(x), \phi^k(y)] = \int d^3x \underbrace{[\Pi^i(x), \phi^k(y)]}_{= -i \delta^{ik} \delta(x-y)} T^a_{ij} \phi^j(x) = -i T^a_{ik} \phi^j(y)$$

$$[Q^a, \Pi^k(y)] = \int d^3x [\Pi^i(x) T^a_{ij} \phi^j(x), \Pi^k(y)] = \int d^3x \Pi^i(x) T^a_{ij} [\phi^j(x), \Pi^k(y)] = i \Pi^i(y) T^a_{ik}$$

$$\begin{aligned} [Q^a, Q^b] &= \int d^3x [Q^a, \Pi^i(x) T^b_{ij} \phi^j(x)] = \int d^3x \Pi^i(x) T^b_{ij} [Q^a, \phi^j(x)] + \int d^3x [Q^a, \Pi^i(x)] T^b_{ij} \phi^j(x) \\ &= -i \int d^3x \Pi^i(x) T^b_{ij} T^a_{jk} \phi^k(x) + i \int d^3x \Pi^k(x) T^a_{ki} T^b_{ij} \phi^j(x) \stackrel{[T^a, T^b] = f^{abc} T^c}{=} \\ &= i \int d^3x \Pi^i(x) [T^a, T^b]_{ij} \phi^j(x) = i f^{abc} Q^c \end{aligned}$$

hence the unitary transformations $U(\vec{E})$ constitute a unitary representation of the symmetry group in the Hilbert space of the theory. The spectrum (masses) of the theory is thus organized in degenerate multiplets corresponding to the irreducible representations of the symmetry.

Let us now consider the case in which spontaneous symmetry breaking (SSB) happens, a phenomenon associated to the presence of degenerate vacua. In this case one could guess that the unitary operator representing the symmetry can be used to pass from a vacuum state to another one, but the situation is more complicated.

Let $|0\rangle$ be a vacuum state which is not invariant under the symmetry, then $Q|0\rangle \neq 0$ and let us define $|\psi\rangle = Q|0\rangle$. We then have (using $Q = \int d^3x J_0(x)$ and translation invariance)

$$\begin{aligned}\langle\psi|\psi\rangle &= \langle 0|Q^2|0\rangle = \int d^3x \langle 0|Q J_0(x)|0\rangle = \int d^3x \langle 0|Q e^{i\vec{p}\cdot\vec{x}} J_0(0) e^{-i\vec{p}\cdot\vec{x}}|0\rangle = \hat{p}|0\rangle = 0 \\ &= \int d^3x \langle 0|e^{i\vec{p}\cdot\vec{x}} Q J_0(0)|0\rangle = \int d^3x \langle 0|Q J_0(0)|0\rangle \text{ which is divergent.}\end{aligned}$$

Hence either $Q|0\rangle = 0$ or $Q|0\rangle$ is not well defined (FABRI-PIGASSO theorem) and the symmetry can not be implemented as a unitary transformation in the Hilbert space.

In QM there cannot be SSB (the fundamental state is always non degenerate, see, e.g. TESCHL

"Mathematical methods in quantum mechanics with applications to Schrödinger operators" § 10.5)

and the physical reason is the following: let us consider for the sake of the simplicity a system with symmetry $P: x \rightarrow -x$ (e.g. the double well $V(x) = g(x^2 - a^2)^2$) and let us imagine

the symmetry to be broken, i.e. the existence of two degenerate fundamental states $|\psi_1\rangle$ and $|\psi_2\rangle$

such that $P|\psi_1\rangle = |\psi_2\rangle$ and $P|\psi_2\rangle = |\psi_1\rangle$ ($[H, P] = 0$ so we have to consider a representation anyway)

But then we can consider the states $|\psi_{\pm}\rangle = \frac{|\psi_1\rangle \pm |\psi_2\rangle}{\sqrt{2}}$ and $P|\psi_{\pm}\rangle = \pm|\psi_{\pm}\rangle$ and

$\langle\psi_{\pm}|H|\psi_{\pm}\rangle = E_0 \pm \text{Re} \langle\psi_1|H|\psi_2\rangle$ so we see that $|\psi_1\rangle$ and $|\psi_2\rangle$ cannot be fundamental states unless $\langle\psi_1|H|\psi_2\rangle$ vanishes. Since the wavefunction of the fundamental state is ≥ 0 this fact is related to the impossibility of strictly localize the wave function, i.e. to the tunnel effect.

In QFT the tunnel effect is exponentially suppressed by the volume, and vanishes in the thermodynamic limit, so we can have SSB.

Let us denote by $|\phi_i\rangle$ the vacuum states (i.e. states which are discrete eigenvectors of the momentum operator with eigenvalue zero), and let $A(\vec{x})$ and $B(\vec{y})$ two local operators. Then

$$\langle\phi_i|A(\vec{x})B(0)|\phi_j\rangle = \sum_k \langle\phi_i|A(0)|\phi_k\rangle \langle\phi_k|B(0)|\phi_j\rangle + \int d^3p \sum_{\alpha} \underbrace{\langle\phi_i|A(0)|\alpha, p\rangle \langle\alpha, p|B(0)|\phi_j\rangle}_{\text{generic continuum state}} e^{-i\vec{p}\cdot\vec{x}}$$

then for $x \rightarrow \infty$ we have (Riemann-Lebesgue lemma)

$$\langle\phi_i|A(\vec{x})B(0)|\phi_j\rangle \xrightarrow{x \rightarrow \infty} \sum_k \langle\phi_i|A(0)|\phi_k\rangle \langle\phi_k|B(0)|\phi_j\rangle \text{ and in the same way we have}$$

$$\langle\phi_i|B(0)A(\vec{x})|\phi_j\rangle \xrightarrow{x \rightarrow \infty} \sum_k \langle\phi_i|B(0)|\phi_k\rangle \langle\phi_k|A(0)|\phi_j\rangle$$

Moreover by consistency we have $[A(\vec{x}), B(\vec{y})] = 0$ if $\vec{x} \neq \vec{y}$ thus the matrices of $A(0)$ and $B(0)$ (and any other local operator) commute with each other, so we can find a basis $|\tilde{\phi}_i\rangle$

such that $\langle\tilde{\phi}_i|A(0)|\tilde{\phi}_j\rangle \propto \delta_{ij}$
 $\uparrow$
 or any local op.

Any local operator (even the ones explicitly breaking the symmetry) will then have

vanishing matrix elements between different $|\tilde{\phi}_i\rangle$ (superselection rule). By adding a symmetry breaking perturbation we however lift the degeneracy, and this enable us "to select" a single state $|\tilde{\phi}_i\rangle$ as the "true" vacuum.

NOTE: vacuum expectation values computed using a single state $|\tilde{\phi}_i\rangle$ satisfies the cluster

property: $\langle \tilde{\phi}_i | A(\bar{x}) B(0) | \tilde{\phi}_i \rangle \xrightarrow{x \rightarrow \infty} \sum_j \langle \tilde{\phi}_i | A(\bar{x}) | \tilde{\phi}_j \rangle \langle \tilde{\phi}_j | B(0) | \tilde{\phi}_i \rangle = \langle \tilde{\phi}_i | A(\bar{x}) | \tilde{\phi}_i \rangle \langle \tilde{\phi}_i | B(0) | \tilde{\phi}_i \rangle$

since everything is diagonal. Cluster property is not satisfied by linear combinations:

$$(\alpha^* \langle \tilde{\phi}_1 | + \beta^* \langle \tilde{\phi}_2 |) A(\bar{x}) B(0) (\alpha | \tilde{\phi}_1 \rangle + \beta | \tilde{\phi}_2 \rangle) \xrightarrow{x \rightarrow \infty} \sum_k (\alpha^* \langle \tilde{\phi}_1 | + \beta^* \langle \tilde{\phi}_2 |) A(\bar{x}) | \tilde{\phi}_k \rangle \langle \tilde{\phi}_k | B(0) | (\alpha | \tilde{\phi}_1 \rangle + \beta | \tilde{\phi}_2 \rangle) \\ = |\alpha|^2 \langle \tilde{\phi}_1 | A(\bar{x}) | \tilde{\phi}_1 \rangle \langle \tilde{\phi}_1 | B(0) | \tilde{\phi}_1 \rangle + |\beta|^2 \langle \tilde{\phi}_2 | A(\bar{x}) | \tilde{\phi}_2 \rangle \langle \tilde{\phi}_2 | B(0) | \tilde{\phi}_2 \rangle$$

$$(\alpha^* \langle \tilde{\phi}_1 | + \beta^* \langle \tilde{\phi}_2 |) A(\bar{x}) (\alpha | \tilde{\phi}_1 \rangle + \beta | \tilde{\phi}_2 \rangle) = |\alpha|^2 \langle \tilde{\phi}_1 | A(\bar{x}) | \tilde{\phi}_1 \rangle + |\beta|^2 \langle \tilde{\phi}_2 | A(\bar{x}) | \tilde{\phi}_2 \rangle$$

and analogously for the matrix element of $B(0)$, hence cluster property is not satisfied unless $\alpha=0$ or $\beta=0$.

NOTE when the symmetry is spontaneously broken all "local" operatorial equations remain valid, but since the states are not invariant under the symmetry the consequences of the symmetry are less manifest. All functional generators computed by path-integral are still invariant under the symmetry, since in the path-integral we are effectively summing on all $|\tilde{\phi}_i\rangle$, unless an explicit symmetry breaking term is added.

For the particularly simple case of the free massless scalar and the SSB of the shift symmetry $\phi(\bar{x}) \rightarrow \phi(\bar{x}) + \lambda$ see ITZYKSON, ZUBER "Quantum field theory" 11-2-2. For a treatise of the mathematical treatment of SSB see STROCCHI "Elements of quantum mechanics of infinite systems"

WARD-TAKAHASHI identities for global symmetries

We now deduce the Ward-Takahashi identities for global symmetries, and we can think for concreteness to the case of the $O(N)$ invariant real scalar ϕ^4 theory with Euclidean Lagrangian density

$$\mathcal{L} = \frac{1}{2} \sum_{\mu} (\partial_{\mu} \vec{\phi}) \cdot (\partial_{\mu} \vec{\phi}) + \frac{1}{2} \mu^2 |\vec{\phi}|^2 + \frac{\lambda}{4!} |\vec{\phi}|^4$$

The generating functional of n -point correlation function is

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \exp \left\{ -S[\phi] + \int J_i(x) \phi_i(x) d^D x \right\}$$

and if we perform the change of integration variable $\vec{\phi} \rightarrow \vec{\phi}' = \vec{\phi} + \delta \vec{\phi}$, with $\delta \phi_i = -\epsilon^a T_{ij}^a \phi_j$,

$$\mathcal{Z}[J] = \int \mathcal{D}\phi' \exp \left\{ -S[\phi'] + \int J_i(x) \phi'_i(x) d^D x \right\} = \int \mathcal{D}\phi \exp \left\{ -S[\phi] + \int J_i \phi_i d^D x - \int J_i \epsilon^a T_{ij}^a \phi_j d^D x \right\}$$

$S[\phi] = S[\phi']$ since $\phi'(x)$ is the infinitesimal variation of $\phi(x)$ under a symmetry
 $\mathcal{D}\phi = \mathcal{D}\phi'$ since the measure is invariant under isometries

and by expanding in ε^a we find

$$0 = \delta Z[\mathcal{J}] = - \int d^D x \left(\int \mathcal{J}^i(x) \varepsilon^a T_{ij}^a \phi^j(x) d^D x \right) e^{-S[\phi] + \int \mathcal{J}_i \phi^i d^D x}$$

hence we get

$$\int d^D x \left(\int \mathcal{J}^i(x) T_{ij}^a \phi^j(x) d^D x \right) e^{-S[\phi] + \int \mathcal{J}_i \phi^i d^D x} = 0$$

which we can rewrite in the form $\left(\int d^D x \mathcal{J}^i(x) T_{ij}^a \frac{\delta}{\delta \mathcal{J}_j(x)} \right) Z[\mathcal{J}] = 0$, which are the W-T identities for $Z[\mathcal{J}]$. Since $W[\mathcal{J}] = \log Z[\mathcal{J}]$ we also have

$$\left(\int d^D x \mathcal{J}^i(x) T_{ij}^a \frac{\delta}{\delta \mathcal{J}_j(x)} \right) W[\mathcal{J}] = 0$$

Expanding these equations in power of $\mathcal{J}$ we get identities that are consequence of the symmetry:

$$Z[\mathcal{J}] = 1 + \int d^D x \mathcal{J}_k(x) \langle \phi_k(x) \rangle d^D x + \frac{1}{2} \int d^D x \mathcal{J}_k(x) \mathcal{J}_l(y) \langle \phi_k(x) \phi_l(y) \rangle d^D x d^D y + \dots$$

thus W-T identities read

$$0 = \int d^D x \mathcal{J}^i(x) T_{ij}^a \langle \phi_j(x) \rangle d^D x + \int d^D x d^D y \mathcal{J}^i(x) T_{ij}^a \mathcal{J}_k(y) \langle \phi_j(x) \phi_k(y) \rangle + \dots$$

From the first term we have $T_{ij}^a \langle \phi_j(x) \rangle = 0$, which means that $\langle \vec{\phi} \rangle$ is invariant under the symmetry and hence, by Schur lemma, $\langle \phi_j \rangle = 0$ (remember that when performing the functional integral we are summing on all degenerate supersselected sector. To select just one of these we need to explicitly break the symmetry). From the second term we find

by applying $\delta / \delta \mathcal{J}^l(x)$

$$\int d^D y T_{lj}^a \mathcal{J}_k(y) \langle \phi_j(x) \phi_k(y) \rangle + \int d^D x \mathcal{J}^i(x) T_{ij}^a \langle \phi_j(x) \phi_l(x) \rangle = 0 \text{ hence}$$

$$T_{lj}^a \langle \phi_j(x) \phi_k(x) \rangle + T_{kj}^a \langle \phi_j(x) \phi_l(x) \rangle = 0 \text{ where } a, l \text{ and } k \text{ are free parameters.}$$

T_{ij}^a is antisymm, while $M_{jk} = \langle \phi_j(\vec{w}) \phi_k(\vec{z}) \rangle \underset{\text{transl.}}{=} \langle \phi_j(0) \phi_k(\vec{z} - \vec{w}) \rangle \underset{\text{parity}}{=} \langle \phi_j(0) \phi_k(\vec{w} - \vec{z}) \rangle = \langle \phi_k(\vec{w}) \phi_j(\vec{z}) \rangle = M_{kj}$

hence we can write $0 = T_{lj}^a M_{jk} + T_{kj}^a M_{jl} = T_{lj}^a M_{jk} - M_{lj} T_{jk}^a = [T^a, M]_{lj}$ and since the T^a generate on irreducible rep. we have by Schur lemma $M \propto \mathbb{1}$, which can be written in the form

$$\langle \phi_i(x) \phi_j(y) \rangle = \frac{\delta_{ij}}{N} \sum_k \langle \phi_k(x) \phi_k(y) \rangle$$

To write the W-T identities for the effective action is sufficient to remember that

from the definition $\varphi_i(x) = \frac{\delta W}{\delta \mathcal{J}_i(x)}$ and $\Gamma[\varphi] + W[\mathcal{J}] = \int \mathcal{J}_i(x) \phi_i(x) d^D x$ we have

$$\mathcal{J}_i(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi_i(x)} \text{ hence } 0 = \int d^D x \mathcal{J}^i(x) T_{ij}^a \frac{\delta W[\mathcal{J}]}{\delta \mathcal{J}_j(x)} = \int d^D x \frac{\delta \Gamma[\varphi]}{\delta \varphi_i(x)} T_{ij}^a \varphi_j(x)$$

$$\int d^D x \frac{\delta \Gamma[\varphi]}{\delta \varphi_i(x)} T_{ij}^a \varphi_j(x) = 0$$

NOTE: if a regularization exists which respect the global symmetry, the W-T identities hold also for the regularized effective action, and since this is true for any value of the regulator parameter (e.g. ϵ in dim. reg.) W-T identities hold also for Γ_{div} , hence the structure of the divergences is constrained by symmetry.

Let us now add to the action a linear symmetry breaking term:

$$S[\phi] = S_{\text{sym}}[\phi] - \int h_i \phi_i(x) d^D x$$

$$\text{where } S_{\text{sym}}[\phi] = \int d^D x \left\{ \frac{1}{2} \sum_i (\partial_\mu \phi_i)^2 + \frac{x}{2} \sum_i \phi_i^2 + \frac{g}{4!} \left(\sum_i \phi_i^2 \right)^2 \right\}.$$

To carry out perturbative computations in this case we have to proceed as follows: if we

denote by $\vec{v}_0$ the static solution of the eq. of motion $\frac{\delta S}{\delta \phi_i} = \frac{\delta S_{\text{sym}}}{\delta \phi_i} - h_i = 0$, which satisfies $(x + \frac{g}{6} v_0^2) \vec{v}_0 = \vec{h}$ ($\vec{v}_0$ is the tree level expectation value of $\vec{\phi}$)

we can perform the change of variable $\vec{\phi} = \vec{v}_0 + \vec{\chi}$, obtaining from

$$\frac{\delta}{\delta \phi_i \delta \phi_j} S = \frac{\delta}{\delta \phi_j} \left(x \phi_i + \frac{g}{3!} \left(\sum_k \phi_k^2 \right) \phi_i \right) = x \delta_{ij} + \frac{g}{3!} \left(\sum_k \phi_k^2 \right) \delta_{ij} + \frac{g}{3} \phi_i \phi_j$$

$$S_{\text{sym}} - \int h_i \phi_i d^D x = \int d^D x \left\{ \frac{1}{2} (\partial_\mu \vec{\chi})^2 + \left[\left(\frac{1}{2} x + \frac{1}{12} g v_0^2 \right) \delta_{ij} + \frac{1}{6} g v_{0i} v_{0j} \right] \chi_i \chi_j + O(\chi^3) \right\}$$

In this expression there is no more a linear term but the mass term is no longer symmetric, so it is not easy to understand the structure of divergences.

To understand the role of the symmetry in this case we can note that

$$Z[\mathcal{J}] = \int \mathcal{D}\phi \exp \left\{ -S_{\text{sym}}[\phi] + \int (\vec{\mathcal{J}}(x) + \vec{h}) \cdot \vec{\phi} d^D x \right\} = \underline{Z_{\text{sym}}[\mathcal{J} + h]}, \text{ and thus also } \underline{W[\mathcal{J}] = W_{\text{sym}}[\mathcal{J} + h]}.$$

From W-T identities for W_{sym}

$$\left(\int d^D x \mathcal{J}^i(x) T_{ij}^a \frac{\delta}{\delta \mathcal{J}_j(x)} \right) W_{\text{sym}}[\mathcal{J}] = 0$$

$$\text{we get, by performing the substitution } \mathcal{J} \rightarrow \mathcal{J} + h: \left(\int d^D x (\mathcal{J}^i(x) + h^i) T_{ij}^a \frac{\delta}{\delta \mathcal{J}_j(x)} \right) W[\mathcal{J}] = 0$$

To understand the structure of divergences it is

however more convenient to use the effective action, defined by

$$\varphi(x) = \frac{\delta W_{\text{sym}}[\mathcal{J} + h]}{\delta \mathcal{J}(x)} \quad \text{and} \quad \Gamma[\varphi] + W_{\text{sym}}[\mathcal{J} + h] = \int \vec{\mathcal{J}} \cdot \vec{\varphi} d^D x \quad \times$$

moreover in the symmetric case we have

$$\varphi_{\text{sym}}(x) = \frac{\delta W_{\text{sym}}[\mathcal{J}]}{\delta \mathcal{J}(x)} \quad \text{and} \quad \Gamma_{\text{sym}}[\varphi_{\text{sym}}] + W_{\text{sym}}[\mathcal{J}] = \int \vec{\mathcal{J}} \cdot \vec{\varphi}_{\text{sym}} d^D x$$

and by replacing in these expression $\mathcal{J}(x) \rightarrow \mathcal{J}(x) + h$ we have

$$\varphi(x) = \frac{\delta W_{\text{sym}}[\mathcal{J} + h]}{\delta \mathcal{J}(x)} \quad \text{and} \quad \Gamma_{\text{sym}}[\varphi] + W_{\text{sym}}[\mathcal{J} + h] = \int \vec{\mathcal{J}} \cdot \vec{\varphi} d^D x + \int \vec{h} \cdot \vec{\varphi} d^D x.$$

By comparison with $\times$ we thus have

$$\Gamma[\varphi] = \Gamma_{\text{sym}}[\varphi] - \int \vec{h} \cdot \vec{\varphi} d^D x$$

from which we see that the linear symmetry breaking term enters $\Gamma[\varphi]$ exactly in the same form as $S[\phi]$. Considering a loop expansion of the regularized theory we reach the conclusion that the linear term does not introduce new classes of divergences and it does renormalize. Note that new divergent diagrams can appear, the point is that the use of the renormalization constants computed at $\vec{h}=0$ is sufficient to make the effective action finite also for $\vec{h} \neq 0$.

From the definition of the effective action we have $\varphi(x) = \frac{\delta W[S]}{\delta S(x)}$ and $\mathcal{S}(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)}$ and if we define $\varphi_i = \varphi_i(x)|_{\vec{S}=0} = \langle \phi_i \rangle$ then $\left. \frac{\delta \Gamma}{\delta \varphi_i} \right|_{\varphi=\varphi} = 0$ and from $\Gamma[\varphi] = \Gamma_{\text{sym}}[\varphi] - \int d^D x \vec{h}_i \cdot \vec{\varphi}$ we have $\left. \frac{\delta \Gamma_{\text{sym}}[\varphi]}{\delta \varphi_i(x)} \right|_{\varphi=\varphi} = h_i$ and $\vec{\varphi} = \vec{\varphi}_0$ in the semiclassical limit.

From the W-T identities of the symmetric case $\int d^D x \frac{\delta \Gamma_{\text{sym}}[\varphi]}{\delta \varphi_i(x)} T_{ij}^a \varphi_j(x) = 0$, using $\Gamma[\varphi] = \Gamma_{\text{sym}}[\varphi] - \int d^D x \vec{h}_i \varphi_i$ and $\vec{\varphi} = \vec{\varphi}_0 + \vec{\eta}$ we get

$$\int d^D x T_{ij}^a \left(\frac{\delta \Gamma[\varphi + \eta]}{\delta \eta_j(x)} + h_i \right) (\varphi_j + \eta_j(x)) = 0$$

which generates the W-T identities for 1PI computed correlation functions when using

$$\Gamma[\varphi] = \sum_{m=1}^{\infty} \frac{1}{m!} \int \left(\prod_{i=1}^m dx_i \right) T_{(m)}^a(x_1, \dots, x_m) \eta(x_1) \dots \eta(x_m) \quad (\text{where } \eta(x) = \varphi(x) - \langle \phi \rangle \text{ and indices are implied})$$

If we consider the lowest order in η (thus $\eta=0$) this gives (since $\left. \frac{\delta \Gamma[\varphi + \eta]}{\delta \eta} \right|_{\eta=0} = 0$)

$$T_{ij}^a h_i \varphi_j = 0$$

which in the $O(N)$ case easily implies $\vec{\varphi} \parallel \vec{h}$: for $N=2$ we only have $T^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T_{ij}^1 h_i \varphi_j = 0$ becomes $h_1 \varphi_2 - h_2 \varphi_1 = 0$, hence $h_1/h_2 = \varphi_1/\varphi_2$ and $\vec{\varphi} \parallel \vec{h}$; in the general case the generators can be written $T_{ij}^{(ke)} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ and the previous argument can be applied to any couple of indices.

To study the next-to-leading order we can differentiate w.r.t. $\eta_k(\vec{z})$ and then fix $\eta=0$: from

$$\left. \frac{\delta \Gamma[\varphi + \eta]}{\delta \eta} \right|_{\eta=0} = 0 \quad \text{we find} \quad \int d^D x \left\{ T_{ij}^a \frac{\delta^2 \Gamma[\varphi + \eta]}{\delta \eta_i(x) \delta \eta_k(\vec{z})} \right\} \varphi_j + T_{ik}^a h_i \delta(x-\vec{z}) = 0 \quad \text{moreover}$$

$$\left. \frac{\delta^2 \Gamma[\varphi + \eta]}{\delta \eta_i(x) \delta \eta_k(\vec{z})} \right|_{\eta=0} = T_{ik}^2(x, \vec{z}) = \int \frac{d^D p}{(2\pi)^D} e^{i(\vec{x}-\vec{z}) \cdot \vec{p}} \tilde{T}_{ik}^{(2)}(p) \quad (\text{see pag. R1 for Fourier transform conventions})$$

thus

$$T_{ij}^a \tilde{T}_{ik}^{(2)}(0) \varphi_j + T_{ik}^a h_i = 0, \quad \text{and using } (T^a)^t = -T^a \text{ we have } \varphi_j T_{ji}^a \tilde{T}_{ik}^{(2)}(0) = h_i T_{ik}^a,$$

which fixes the structure of $\tilde{T}_{ik}^{(2)}(0)$ for $h \neq 0$: at zero momentum the tensor structure

of $\tilde{T}_{ik}(p)$ can only depend on $\vec{h}$, and we can distinguish longitudinal and transverse components.

The longitudinal component is not constrained since $\vec{v} \parallel \vec{h}$, thus $v_j T_{ji} h_i = 0$, while for the transverse one we have $\tilde{T}_{ik}^{(2,T)}(p) = \tilde{T}_{ij}^{(2)}(p) P_{jk}^T$ and $P_{ij}^T = \delta_{ij} - \frac{h_i h_j}{h^2}$, moreover

$$\tilde{T}_{ik}^{(2,T)}(0) = \tilde{T}_T(0) P_{ik}^T \text{ hence}$$

$$v_j T_{ji} \underbrace{\tilde{T}_{ik}^{(2)}(0) P_{ke}^T}_{\tilde{T}_T(0) P_{ik}^T} = h_i T_{ik}^0 P_{ke}^T$$

$$\text{and thus } \tilde{T}_T(0) v_j T_{ji} P_{ik}^T = h_j T_{ji} P_{ik}^T. \text{ Since we already}$$

know that $\vec{v} \parallel \vec{h}$ we conclude that $\tilde{T}_T(0) = |\vec{h}|/|\vec{v}|$

SSB and the Goldstone theorem

Spontaneous symmetry breaking emerges in the limit $|\vec{h}| \rightarrow 0$ (in the thermodynamic limit, which has always been implied in the previous discussion): in this limit the action becomes symmetric, but depending on the values of the parameters the spectrum may or may not be symmetric.

$$\mathcal{L}_E = \frac{1}{2} \sum_{ni} (\partial_\mu \phi_i)^2 + V(\vec{\phi}) \quad \text{with} \quad V(\phi_i) = \frac{1}{2} x \sum_i \phi_i^2 + \frac{1}{4!} g (\sum_i \phi_i^2)^2 - \sum_i h_i \phi_i$$

In the semiclassical limit $V_{\text{eff}}[\varphi] \simeq V[\varphi]$ (there are in fact subtleties: $V[\varphi]$ is always convex since $\tilde{W}^{(2)} \geq 0$ due to the Cauchy-Schwarz inequality, hence $\tilde{T}^{(2)} \geq 0$) and $\left. \frac{\delta \Gamma}{\delta \varphi_i} \right|_{\varphi=\vec{v}} = 0$

* gives $(x + \frac{g}{6} v_0^2) \vec{v}_0 = \vec{h}$ ($\vec{v}_0$ is the tree level approx of $\vec{v}$) and

for $h=0$ the minimum solutions are

$$\text{for } x \geq 0: \quad v_0^2 = 0 \quad \begin{array}{c} \uparrow V(\phi) \\ \text{graph of } V(\phi) \text{ for } x \geq 0, h=0 \end{array}$$

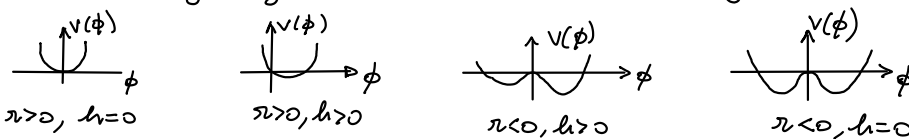
$$\text{for } x < 0: \quad v_0^2 = -\frac{6x}{g} \quad \begin{array}{c} \uparrow V(\phi) \\ \text{graph of } V(\phi) \text{ for } x < 0, h=0 \end{array}$$

non analytic behavior of v_0 at $x=0$.

For $x < 0$ and $h=0$ the classical minimum is degenerate, and starting from one minimum $\vec{v}_0$ we can reach other minima by acting with the symmetry group.

When $h \neq 0$, the position of the minimum $v_0(x, h)$ is analytic, and using this fact

we can analytically connect the $x < 0$ and $x > 0$ regions:



Using $\vec{\phi} = \vec{v}_0 + \vec{\chi}$ we have

$$\mathcal{L}_E = \frac{1}{2} (\partial_\mu \vec{\chi})^2 + \left[\left(\frac{1}{2} x + \frac{1}{12} g v_0^2 \right) \delta_{ij} + \frac{1}{6} g v_{0i} v_{0j} \right] \chi_i \chi_j + O(\chi^3)$$

which for $h=0$ becomes (see * at middle page)

$$\mathcal{L}_E = \frac{1}{2} (\partial_\mu \vec{\chi})^2 + \frac{1}{6} g \varphi_{0i} \varphi_{0j} \chi_i \chi_j + \mathcal{O}(\chi^3)$$

hence the mass matrix is $M_{ij} = \frac{1}{3} g \varphi_{0i} \varphi_{0j}$, which has several zero eigenvalues, corresponding to vectors orthogonal to $\vec{v}$: if $\vec{w} \cdot \vec{v} = 0$ then $M_{ij} w_j = 0$. The important point to be understood is whether this property is preserved at higher orders or not (for a 1-loop computation see CHENG, LI "Gauge theory of elementary particle physics" § 6.3).

Let us remind the W-T identities for the effective action

$$\int d^D x T_{ij}^a \left(\frac{\delta \Gamma[\varphi + \eta]}{\delta \eta_i(x)} + h_i \right) (\varphi_j + \eta_j) = 0 \quad \text{where } \vec{v} \text{ is defined in such a way that } \left. \frac{\delta \Gamma[\varphi + \eta]}{\delta \eta_i(x)} \right|_{\eta=0} = 0$$

moreover we proved $h_i T_{ij}^a \varphi_j = 0$ (hence $\vec{v} \parallel \vec{h}$), and the SSB condition is $\vec{v} \neq 0$ for $h=0$.

For $h \rightarrow 0$ we have

$$\int d^D x T_{ij}^a \frac{\delta \Gamma[\varphi + \eta]}{\delta \eta_i(x)} (\varphi_j + \eta_j(x)) = 0$$

and to study the $\eta \rightarrow 0$ behavior we have to distinguish two cases:

1) $\vec{v} = 0$ (no SSB) the leading term is

$$\int d^D x T_{ij}^a \frac{\delta^2 \Gamma[\eta]}{\delta \eta_i(x) \delta \eta_j(z)} \bigg|_{\eta=0} \eta_k(z) \eta_j(x) = 0$$

hence using $\frac{\delta^2 \Gamma[\eta]}{\delta \eta_i(x) \delta \eta_j(z)} \bigg|_{\eta=0} = \Gamma_{ik}^2(x, z) = \int \frac{d^D p}{(2\pi)^D} e^{i(\vec{x}-\vec{z}) \cdot \vec{p}} \tilde{\Gamma}_{ik}^{(2)}(p)$ (see R1 for conventions on F.T.)

we get by taking two derivatives $T_{il}^a \tilde{\Gamma}_{im}^{(2)}(0) + T_{im}^a \tilde{\Gamma}_{il}^{(2)}(0) = 0$, which using the symmetry of $\tilde{\Gamma}_{ij}^{(2)}(0)$ and $(T^a)^t = -T^a$ can be written as $[T^a, \tilde{\Gamma}^{(2)}] = 0$, hence by Schur lemma $\tilde{\Gamma}_{ij}^{(2)}(0) = m^2 \delta_{ij}$ and the symmetry is manifest in the spectrum.

2) $\vec{v} \neq 0$ (SSB) the first term of the W-T identities is

$$\int d^D x T_{ij}^a \frac{\delta^2 \Gamma[\varphi + \eta]}{\delta \eta_i(x) \delta \eta_j(z)} \bigg|_{\eta=0} \eta_k(z) \varphi_j = 0 \quad \text{and using } (T^a)^t = -T^a \text{ we have } \boxed{\varphi_j T_{ji}^a \tilde{\Gamma}_{ik}^{(2)}(0) = 0}$$

Let us denote by H the subgroup of G composed by transformations which leave $\vec{v}$ invariant (little group or stabilizer), if $G = O(3)$ then $H = O(2)$. H is generated by T^a such that $T_{ij}^a \varphi_j = 0$ and let us denote by G/H the coset generated by the remaining generators (we sort generators in such a way that H is generated by T^a with $a=1, \dots, \dim H$).

Then $\omega_j^a = \varphi_i T_{ij}^a \neq 0$ for $a = (\dim H) + 1, \dots, \dim G$ and $\omega_j^a \tilde{\Gamma}_{jk}^{(2)}(0) = 0$ for these values, hence a zero eigenvalue (i.e. a massless excitation, a Goldstone boson) is

associated to any "broken" generator (GOLDSTONE theorem). Eg for the $O(N) \rightarrow O(N-1)$

breaking we have $\dim O(N) - \dim O(N-1) = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1$ Goldstone bosons

(note: for $N=1$ we have $\mathbb{Z}_2 \rightarrow \phi$, the symmetry is discrete and no Goldstones are present).

For a different proof of the Goldstone theorem see WEINBERG "The quantum theory of fields" vol. II §19.2-3, for a particularly compact proof see ITZYKSON, ZUBER "Quantum field theory" §11-2-2, for a discussion of some formal aspects and for the nonrelativistic case see STRACCHI "Elements of quantum mechanics of infinite systems" part C.1.

WARD-TAKAHASHI identities in QED

The Euclidean Lagrangian density is $\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(\not{\partial} + m + ie\not{A})\psi + \frac{1}{2} (\partial_\mu A_\mu)^2$, and gauge transformations are $\psi(x) \rightarrow \psi'(x) = e^{i\Lambda(x)}\psi(x)$, $A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \Lambda$, $\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = e^{-i\Lambda(x)}\bar{\psi}(x)$, whose infinitesimal forms are $\delta\psi(x) = i\Lambda(x)\psi(x)$, $\delta\bar{\psi}(x) = -i\Lambda(x)\bar{\psi}(x)$, $\delta A_\mu(x) = -\frac{1}{e} \partial_\mu \Lambda(x)$

The generating functional of n -point correlation function is

$$Z[\zeta, \bar{\eta}, \eta] = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ - \int \mathcal{L} d^Dx + \int (A_\mu \zeta^\mu + \bar{\eta} \psi + \bar{\psi} \eta) d^Dx \right\}$$

By performing the change of variable $\psi \rightarrow \psi'$, $\bar{\psi} \rightarrow \bar{\psi}'$ and $A_\mu \rightarrow A'_\mu$, where primed variables correspond to a gauge transformation, $\mathcal{D}A$ is invariant since the gauge transformation is a functional translation, and $\mathcal{D}\bar{\psi} \mathcal{D}\psi$ is invariant. Hence we find

$$0 = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \delta \left[\int \mathcal{L} d^Dx - \int (A_\mu \zeta^\mu + \bar{\eta} \psi + \bar{\psi} \eta) d^Dx \right] \exp \left\{ - \int \mathcal{L} d^Dx + \int (A_\mu \zeta^\mu + \bar{\eta} \psi + \bar{\psi} \eta) d^Dx \right\}$$

moreover

$$\begin{aligned} \delta \left[\int \mathcal{L} d^Dx - \int (A_\mu \zeta^\mu + \bar{\eta} \psi + \bar{\psi} \eta) d^Dx \right] &= -\frac{1}{e} \int (\partial_\mu A_\mu) \partial^2 \Lambda d^Dx + \frac{1}{e} \int \partial_\mu \partial_\mu \Lambda d^Dx - \int \bar{\eta} i\Lambda \psi d^Dx + \int i\Lambda \bar{\psi} \eta d^Dx = \\ &= \frac{1}{e} \int \Lambda(x) \left[-\frac{1}{3} \partial^2 \partial_\mu A_\mu - \partial_\mu \zeta_\mu - ie(\bar{\eta} \psi - \bar{\psi} \eta) \right] d^Dx \end{aligned}$$

and since $\Lambda(x)$ is an arbitrary function we find

$$\begin{aligned} 0 &= \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \left[-\frac{1}{3} \partial^2 \partial_\mu A_\mu - \partial_\mu \zeta_\mu - ie(\bar{\eta} \psi - \bar{\psi} \eta) \right] \exp \left\{ - \int \mathcal{L} d^Dx + \int (A_\mu \zeta^\mu + \bar{\eta} \psi + \bar{\psi} \eta) d^Dx \right\} = \\ &= \left\{ -\frac{1}{3} \partial^2 \partial_\mu \frac{\delta}{\delta \zeta_\mu(x)} - \partial_\mu \zeta_\mu - ie \left(\bar{\eta}(x) \frac{\delta}{\delta \bar{\eta}(x)} - \eta(x) \frac{\delta}{\delta \eta(x)} \right) \right\} Z[\zeta, \bar{\eta}, \eta] \end{aligned}$$

and using $W[\zeta, \bar{\eta}, \eta] = \log Z[\zeta, \bar{\eta}, \eta]$ we have

$$\left\{ -\frac{1}{3} \partial^2 \partial_\mu \frac{\delta}{\delta \zeta_\mu(x)} - ie \left(\bar{\eta}(x) \frac{\delta}{\delta \bar{\eta}(x)} - \eta(x) \frac{\delta}{\delta \eta(x)} \right) \right\} W[\zeta, \bar{\eta}, \eta] = \partial_\mu \zeta_\mu$$

W-T identities for connected correlation functions.

Using the W-T identities it is simple to prove at all orders of perturbation theory a relation we have seen to be valid at one loop: by applying $\frac{\delta}{\delta \zeta_\nu(y)}$, then fixing to zero the sources we find (note that $\langle A_\nu(x) \psi(x) \rangle = 0$ due to Lorentz invariance)

$$-\frac{1}{3} \partial^2 \partial_\mu^{(x)} G_{\mu\nu}^{(AA)}(x, y) = \partial_\nu^{(x)} \delta(x-y) \quad \text{and in Fourier space} \quad \frac{1}{3} k^2 k_\mu \tilde{G}_{\mu\nu}^{(AA)}(k) = k_\nu$$

i.e. $\underline{k_\mu \tilde{G}_{\mu\nu}^{(AA)}(k) = \frac{1}{3} k_\nu / k^2}$

Remembering that the photon propagator is given by

$$\tilde{D}_{\mu\nu}(k) = \frac{1}{k^2} [\delta_{\mu\nu} - (1-\xi)k_\mu k_\nu / k^2] = \frac{1}{k^2} (\delta_{\mu\nu} - k_\mu k_\nu / k^2) + \xi k_\mu k_\nu / k^4$$

we can write $\tilde{G}_{\mu\nu}^{(AA)}(k) = \tilde{D}_{\mu\nu}(k) + \tilde{R}_{\mu\nu}(k)$, where the last term is the sum of loops corrections. Since $k_\mu \tilde{D}_{\mu\nu}(k) = \xi k_\nu / k^2$ we then have $k_\mu \tilde{R}_{\mu\nu}(k) = 0$, i.e. all loop contributions (and thus all divergences) are transverse.

Let us now consider the effective action:

$$A_\mu(x) = \frac{\delta W[\mathcal{S}, \bar{\eta}, \eta]}{\delta \mathcal{J}_\mu(x)}, \quad \bar{\psi}(x) = -\frac{\delta W[\mathcal{S}, \bar{\eta}, \eta]}{\delta \eta(x)}, \quad \psi(x) = \frac{\delta W[\mathcal{S}, \bar{\eta}, \eta]}{\delta \bar{\eta}(x)}$$

$$\Gamma[A, \bar{\psi}, \psi] + W[\mathcal{S}, \bar{\eta}, \eta] = \int (A_\mu(x) \mathcal{J}_\mu(x) + \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x)) d^D x$$

As in the scalar case we have

$$\frac{\delta \Gamma}{\delta A_\mu(x)} = \mathcal{J}_\mu(x) + \int \frac{\delta \mathcal{J}_\nu(y)}{\delta A_\mu(x)} \frac{\delta}{\delta \mathcal{J}_\nu(y)} \left[\int A_\alpha \mathcal{J}_\alpha - W[\mathcal{S}, \bar{\eta}, \eta] \right]_{A=\text{const}} = \mathcal{J}_\mu(x)$$

and analogously, using the Grassmann chain rule ($\frac{\partial}{\partial \theta} F(\eta, \psi) = \frac{\partial \eta}{\partial \theta} \frac{\partial}{\partial \eta} F + \frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial \psi} F$), we have

$$\frac{\delta \Gamma}{\delta \bar{\psi}(x)} = \eta(x) + \int \frac{\delta \eta(y)}{\delta \bar{\psi}(x)} \frac{\delta}{\delta \eta(y)} \left[\int \bar{\psi} \eta - W[\mathcal{S}, \bar{\eta}, \eta] \right]_{\bar{\psi}=\text{const}} = \eta(x)$$

$$\frac{\delta \Gamma}{\delta \psi(x)} = -\bar{\eta}(x) + \int \frac{\delta \bar{\eta}(y)}{\delta \psi(x)} \frac{\delta}{\delta \bar{\eta}(y)} \left[\int \bar{\eta} \psi - W[\mathcal{S}, \bar{\eta}, \eta] \right]_{\psi=\text{const}} = -\bar{\eta}(x)$$

hence in the W-T identity we have

$$\left(-\frac{1}{\xi} \partial^\alpha \partial_\mu \frac{\delta}{\delta \mathcal{J}_\mu(x)} - i e \left(\underbrace{\bar{\eta}(x) \frac{\delta}{\delta \bar{\eta}(x)}}_{-\delta\Gamma/\delta\psi} - \underbrace{\eta(x) \frac{\delta}{\delta \eta(x)}}_{\delta\Gamma/\delta\bar{\psi}} \right) \right) W[\mathcal{S}, \bar{\eta}, \eta] = \partial_\mu \mathcal{J}_\mu(x) \quad \uparrow \frac{\delta\Gamma}{\delta A_\mu}$$

hence

$$\boxed{\frac{1}{\xi} \partial^\alpha \partial_\mu A_\mu(x) + i e \left(\psi(x) \frac{\delta\Gamma}{\delta \psi(x)} - \bar{\psi}(x) \frac{\delta\Gamma}{\delta \bar{\psi}(x)} \right) + \partial_\mu \frac{\delta\Gamma}{\delta A_\mu} = 0} \quad \text{W-T identities for } \Gamma.$$

non homogeneous term

The homogeneous terms of the W-T. originate from the gauge invariant part of the action, and their sum vanishes for a gauge invariant theory.

Let us explicitly check that the classical action S satisfies the W.T. identities:

$$S = \int \left\{ \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \bar{\psi} (\not{\partial} + m + i e \not{A}) \psi \right\} d^D x =$$

$$= \int \left\{ \frac{1}{2} A_\mu (\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2) A_\nu - \frac{1}{2\xi} A_\mu \partial_\mu \partial_\nu A_\nu + \bar{\psi} (\not{\partial} + m + i e \not{A}) \psi \right\} d^D x$$

hence we have

$$i e \left(\psi \frac{\delta S}{\delta \psi} - \bar{\psi} \frac{\delta S}{\delta \bar{\psi}} \right) + \partial_\mu \frac{\delta S}{\delta A_\mu} = i e \left\{ \psi_i (-i) \bar{\psi}_j (-\not{\partial} + m + i e \not{A})_{ji} - \bar{\psi} (\not{\partial} + m + i e \not{A}) \psi \right\} +$$

$$+ \partial_\mu \left\{ i e \bar{\psi} \partial_\mu \psi + (\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2) A_\nu - \frac{1}{\xi} \partial_\mu \partial_\nu A_\nu \right\} =$$

$$= i e \left\{ \bar{\psi}_j (\not{\partial} + m + i e \not{A})_{ji} \psi_i - \partial_\mu (\bar{\psi} \partial_\mu \psi) - \bar{\psi} (\not{\partial} + m + i e \not{A}) \psi \right\} + i e \partial_\mu (\bar{\psi} \gamma_\mu \psi) - \frac{1}{\xi} \partial^2 \partial_\nu A_\nu = -\frac{1}{\xi} \partial^2 \partial_\nu A_\nu$$

and S satisfies the W.T. identities. By writing $\Gamma = S + \sum_n \hbar^n \Gamma_n$ we have

$$\left\{ i e \left(\psi(x) \frac{\delta}{\delta \psi(x)} - \bar{\psi}(x) \frac{\delta}{\delta \bar{\psi}(x)} \right) + \partial_\mu \frac{\delta}{\delta A_\mu(x)} \right\} \Gamma_n = 0$$

and, in particular the same equation has to be satisfied by the divergent part of Γ_n , since this equation holds for any value of the regularization parameter (obviously assuming a gauge invariant regulator, like, e.g. dim. reg.), hence the divergent terms are gauge invariant at any order of perturbation theory. The general solution of the W.T. identities can in fact be written in the form

$$\Gamma = \Gamma_{\text{gauge inv}} + \frac{1}{2\xi} \int d^D x (\partial_\mu A_\mu)^2$$

and something similar happens also if we add a mass term to the gauge field, see SWIN-SUSTIN "Quantum field theory and critical phenomena" § 18.6

If we differentiate the W.T. identity once w.r.t. $A_\nu(y)$ and we set the fields to zero we get

$$\frac{1}{\xi} \partial_\nu \delta(x-y) + \partial_\mu \frac{\delta \Gamma}{\delta A_\mu(x) \delta A_\nu(y)} = 0 \quad \text{and going in momentum space } k_\mu \tilde{\Gamma}_{\mu\nu}^{(AA)}(k) = \frac{1}{\xi} k^2 k_\nu$$

Since $\tilde{G}_{\mu\nu}^{(AA)}(p) \tilde{\Gamma}_{\nu\rho}^{(AA)}(p) = \delta_{\mu\rho}$, this equation is equivalent to $k_\mu \tilde{G}_{\mu\nu}^{(AA)}(k) = \xi k_\nu / k^2$ found before.

If we consider at least two derivatives w.r.t. A of the W.T. identity, the non homogeneous term disappear, and we immediately find $k_{\mu_1} \tilde{\Gamma}_{\mu_1 \dots \mu_n}^{(A_1 \dots A_n)}(k_1, \dots, k_n) = 0$ if $n > 2$ (obviously all cases with odd n vanishes due to FURRY theorem). For $n=4$ this fact implies that irreducible amputated diagrams of the form  (which have sup. degree of divergence $D_G = 0$) are not divergent, since for each external leg we can factor a term $k_\nu \gamma_{\mu\nu}$ where $\gamma_{\mu\nu}$ is antisymmetric. The effective degree of divergence is thus -4 and the diagram is convergent.

Let us now consider the W.T. identity for the proper vertex $\Gamma^{(A\bar{\psi}\psi)}$: by differentiating the W.T. id.

w.r.t. $\psi(y)$ and then to $\bar{\psi}(z)$, then finally considering vanishing sources we find

$$0 = \frac{\delta}{\delta \bar{\psi}(x_2)} \frac{\delta}{\delta \psi(x_3)} \left[\partial_\mu^{(1)} \frac{\delta \Gamma}{\delta A_\mu(x_1)} + i e \left(\psi(x_1) \frac{\delta \Gamma}{\delta \psi(x_1)} - \bar{\psi}(x_1) \frac{\delta \Gamma}{\delta \bar{\psi}(x_1)} \right) \right] \Big|_{A=\bar{\psi}=\psi=0} =$$

$$= \partial_\mu^{(1)} \Gamma_\mu^{(A\bar{\psi}\psi)}(x_1, x_2, x_3) + i e \delta(x_1 - x_3) \Gamma^{(\bar{\psi}\psi)}(x_2, x_1) - i e \delta(x_1 - x_2) \Gamma^{(\bar{\psi}\psi)}(x_1, x_3)$$

multiplying by $\prod_{i=1}^3 e^{i k_i x_i}$ and integrating in $\prod_{i=1}^3 d^D x_i$ we get, remembering

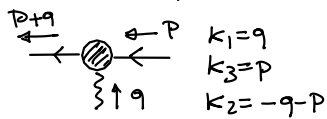
$$(2\pi)^D \delta^D(\sum_k p_k) \tilde{\Gamma}_\mu^{(A\bar{\psi}\psi)}(p_1, \dots, p_n) = \int \Gamma_\mu(x_1 \dots x_n) \prod_{i=1}^n d^D x_i e^{i x_i p_i}$$

the identity

$$-i k_{3\mu} (2\pi)^D \delta(\sum k_i) \tilde{\Gamma}_\mu^{(A\bar{\psi}\psi)}(k_1, k_2, k_3) + i e (2\pi)^D \delta(\sum k_i) \tilde{\Gamma}^{(\bar{\psi}\psi)}(k_2, k_3 + k_1) - i e (2\pi)^D \delta(\sum k_i) \tilde{\Gamma}^{(\bar{\psi}\psi)}(k_1 + k_2, k_3)$$

from which

$$k_{1\mu} \tilde{\Gamma}_{\mu}^{(A\bar{\Psi}\Psi)}(k_1, k_2, k_3) = e \tilde{\Gamma}^{(\bar{\Psi}\Psi)}(k_2, k_3+k_1) - e \tilde{\Gamma}^{(\bar{\Psi}\Psi)}(k_1+k_2, k_3) \quad \text{where } \sum k_i = 0$$



$$q_{\mu} \tilde{\Gamma}_{\mu}^{(A\bar{\Psi}\Psi)}(q, -q-p, p) = \frac{e}{\tilde{W}^{(\bar{\Psi}\Psi)}(-q-p, q+p)} - \frac{e}{\tilde{W}^{(\bar{\Psi}\Psi)}(-p, p)}$$

and in particular $e \frac{\partial}{\partial p_{\mu}} \left(\frac{1}{\tilde{W}^{(\bar{\Psi}\Psi)}(-p, p)} \right) = \tilde{\Gamma}_{\mu}^{(A\bar{\Psi}\Psi)}(0, -p, p)$, which is the "original" Ward identity.

Note that at tree level $\tilde{W}^{(\bar{\Psi}\Psi)}(-p, p) = \frac{1}{i \not{p} + m}$ hence $e \frac{\partial}{\partial p_{\mu}} \left(\frac{1}{\tilde{W}^{(\bar{\Psi}\Psi)}(-p, p)} \right) = i e \gamma^{\mu}$, which is the vertex at tree level

(remember the minus sign between $\tilde{T}_3$ and $\tilde{W}_{3, \text{amp}}$, see PSF12)

QED RENORMALIZATION AND RG

The QED bare Euclidean Lagrangian is $\mathcal{L}_{\text{QED}, B}^E = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} \bar{\Psi}_B (\not{\partial} + m_B + i e_B \not{A}_B) \Psi_B$
 moreover let us remember that (see R10) $[e] = \frac{4-D}{2}$ hence $[e] = \epsilon/2$ using $\epsilon = 4-D$

The renormalization constants are defined by

$$A_B = Z_A^{1/2} A, \quad \Psi_B = Z_{\Psi}^{1/2} \Psi, \quad m_B = Z_m m \quad (\text{in MS scheme!}), \quad e_B = \mu^{\epsilon/2} Z_e^{1/2} e, \quad \bar{\Psi}_B = Z_{\bar{\Psi}} \bar{\Psi}$$

$$\alpha = \frac{e^2}{4\pi}, \quad \alpha_B = \mu^{\epsilon} Z_e \alpha$$

hence the renormalized Lagrangian is

$$\mathcal{L}_{\text{QED}}^E = \frac{1}{4} Z_A F_{\mu\nu}^2 + \frac{Z_A}{Z_3} \frac{1}{2} (\partial_{\mu} A_{\mu})^2 + Z_{\Psi} \bar{\Psi} (\not{\partial} + Z_m m + i Z_e^{1/2} Z_A^{1/2} e \not{A}) \Psi$$

We have seen that the general solution of the W.T. identities for the effective action

$$\Gamma = \Gamma_{\text{gauge inv}} + \frac{1}{2} \bar{\xi} (\partial_{\mu} A_{\mu})^2 \quad \text{hence we have in particular}$$

1) all divergent contributions to the effective action are gauge invariant

$$2) \quad \boxed{Z_A/Z_3 = 1}$$

From 1) it follows that the covariant derivative has to be invariant under renormalization

$$D_{\mu} = \partial_{\mu} + i e_B A_{\mu} = \partial_{\mu} + i e A_{\mu} \quad \text{hence } \boxed{Z_e Z_A = 1}, \text{ a fact that can be proven more}$$

directly using $e \frac{\partial}{\partial p_{\mu}} (1/\tilde{W}^{(\bar{\Psi}\Psi)}(-p, p)) = \tilde{\Gamma}_{\mu}^{(A\bar{\Psi}\Psi)}(0, -p, p)$ since $e/\tilde{W}^{(\bar{\Psi}\Psi)} \rightarrow \frac{Z_e^{1/2}}{Z_{\Psi}} \frac{e}{\tilde{W}^{(\bar{\Psi}\Psi)}}$

$$\text{and } \tilde{\Gamma}_{\mu}^{(A\bar{\Psi}\Psi)} \rightarrow \frac{1}{Z_A^{1/2} Z_{\Psi}} \tilde{\Gamma}_{\mu}^{(A\bar{\Psi}\Psi)}.$$

NOTE: the fact that divergences can be absorbed by redefining the gauge invariant terms of the action does not imply that renormalization constants are gauge invariant.

However in MS scheme only Z_{Ψ} depends on $\bar{\xi}$ (see ZINN-JUSTIN "Anomalous field theory and critical phenomena" §18.4 for the explicit form of the $\bar{\xi}$ -dependence)

It is not difficult to see that in HS z_e is independent of ξ : since HS is a mass-independent scheme we have $z_e = 1 + \frac{a_1(e, \xi)}{\epsilon} + \frac{a_2(e, \xi)}{\epsilon^2} + \dots$ and since e_B^2 is independent of ξ we have $0 = \frac{d}{d\xi} (e^2 z_e) = \frac{de^2}{d\xi} + \frac{1}{\epsilon} \left(\frac{de^2}{d\xi} a_1 + e^2 \frac{da_1}{d\xi} \right) + \frac{1}{\epsilon^2} \left(\frac{de^2}{d\xi} a_2 + e^2 \frac{da_2}{d\xi} \right) + \dots$
 hence $\frac{de^2}{d\xi} = 0$, $\frac{da_m}{d\xi} = 0$ and thus z_e is independent of ξ . In an analogous way we see that also z_m is independent of ξ .

To obtain the RG equations in HS scheme for the connected 1PI amputated correlation functions we start as usual from

$$\Gamma_B^{(\ell A, m \bar{\psi} \psi)}(k_i, \alpha_B, \xi_B, m_B) = z_A^{-\ell/2} z_\psi^{-m} \Gamma^{(\ell A, m \bar{\psi} \psi)}(k_i, \alpha, \xi, m, \mu)$$

and impose $\mu \frac{d}{d\mu} \Gamma_B|_B = 0$. The final equation is

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \delta(\alpha) \xi \frac{\partial}{\partial \xi} + \eta_m(\alpha) m \frac{\partial}{\partial m} - \frac{\ell}{2} \eta_A(\alpha) - m \eta_\psi(\alpha) \right] \Gamma^{(\ell A, m \bar{\psi} \psi)}(k_i, \alpha, \xi, m, \mu) = 0$$

where we defined $\beta(\alpha) = \mu \frac{d}{d\mu} \alpha|_B$, $\eta_A(\alpha) = \mu \frac{d}{d\mu} \log z_A(\alpha)$ and from $z_e z_A = 1$ we have $\alpha = \mu^{-\epsilon} z_e^{-1} \alpha_B = \mu^{-\epsilon} z_A \alpha_B$, thus

$$\beta(\alpha) = \mu \frac{d}{d\mu} \alpha|_B = \mu \frac{d}{d\mu} (\mu^{-\epsilon} z_A \alpha_B) = -\epsilon \alpha + \alpha \frac{1}{z_A} \mu \frac{d}{d\mu} z_A|_B = -\epsilon \alpha + \alpha \eta_A(\alpha)$$

and in particular $\eta_A(\alpha^*) = \epsilon$ if $\beta(\alpha^*) = 0$ and $\alpha^* \neq 0$.

$$\eta_m(\alpha) = -\mu \frac{d}{d\mu} \log z_m \quad \text{so that} \quad \mu \frac{d}{d\mu} m|_B = \mu \frac{d}{d\mu} (m_B z_m^{-1})|_B = -\frac{m_B}{z_m^2} \mu \frac{d}{d\mu} z_m|_B = m \eta_m(\alpha)$$

and analogously $z_3 = z_A$

$$\delta(\alpha) = -\mu \frac{d}{d\mu} \log z_\xi = -\eta_A(\alpha) \quad \text{so that} \quad \mu \frac{d}{d\mu} \xi|_B = \mu \frac{d}{d\mu} (\xi_B z_\xi^{-1})|_B = -\frac{\xi_B}{z_\xi^2} \mu \frac{d}{d\mu} z_\xi|_B = \xi \delta(\alpha)$$

Finally $\eta_\psi = \mu \frac{d}{d\mu} \log z_\psi$. Since $z_\xi = z_A = z_e^{-1}$ and in HS z_e and z_m are independent of ξ all RG functions in HS are independent of ξ but for η_ψ and it can be shown that $\eta_\psi(\alpha, \xi) = \eta_\psi(\alpha, 0) - \frac{\alpha \xi}{2\pi}$ (see ZINN-SUSTIN "Quantum field theory and critical phenomena" § 18.8).

Let us compute z_A to one loop (and hence also z_e): the one loop correction to the photon two point function is  which comes with a minus sign due to the fermion loop

A direct check of the minus sign can be obtained as follows: if we are only interested in photon correlation functions we can fix $\bar{\eta} = \eta = 0$ in $Z[\xi, \bar{\eta}, \eta]$ and integrate out the fermions by using $\int \mathcal{D}(\bar{\psi}, \psi) e^{-\int \bar{\psi} H \psi d^4x} = \det H = \exp\{\text{tr} \log H\}$ hence

$$Z[\xi] = \int \mathcal{D}A \exp\{-S'[A] + \int \bar{J}_\mu A_\mu d^4x\} \quad \text{and}$$

$$S'[A] = \int \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 \right) d^4x - \text{tr} \log(\not{D} + m + i\epsilon \not{A})$$

moreover $\text{tr} \log(\not{p} + m + i\epsilon \not{A}) = \text{tr} \log \{ (\not{p} + m) [1 + (\not{p} + m)^{-1} i\epsilon \not{A}] \} =$

$$= \text{indip of } A + \text{tr} \log [1 + (\not{p} + m)^{-1} i\epsilon \not{A}] = \text{tr} \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

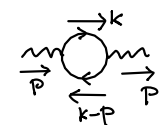
$$= \text{indip of } A + \text{loop diagrams} + \dots \quad (\text{note: loops do not come from fermion contractions, so there is no "-" sign})$$

hence

$$e^{\text{tr} \log [1 + (\not{p} + m)^{-1} i\epsilon \not{A}]} = \text{indip of } A + (\text{loop diagrams}) + \frac{1}{2} (\text{loop diagrams})^2 + \dots$$

and if we consider just the two point function we get

$$\text{two-point function} = \text{loop diagrams} + \dots$$

We now compute the (unperturbed version of the) diagram  (with the "-" sign)

$$\Pi_{\mu\nu}(p) = -(-ie)^2 \int \frac{d^D k}{(2\pi)^D} \frac{\text{tr} [\gamma_\mu (i\not{k} + m) \gamma_\nu (i\not{k} - \not{p} + m)]}{(k^2 + m^2)((k-p)^2 + m^2)}$$

We could simplify a little the computation by using the transversality condition, that we know already, but let us see how it emerges from the computation

γ matrices traces: from $\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\delta^{\alpha\beta}$ and $\text{tr} \delta^{\alpha\beta} = 4\delta^{\alpha\beta}$ we have $\text{tr} \gamma^\alpha \gamma^\beta = 4\delta^{\alpha\beta}$

$$\begin{aligned} \text{moreover } \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\epsilon) &= 2\delta^{\alpha\beta} \text{tr}(\gamma^\delta \gamma^\epsilon) - \text{tr}(\gamma^\beta \gamma^\alpha \gamma^\delta \gamma^\epsilon) = \\ &= 8\delta^{\alpha\beta} \delta^{\delta\epsilon} - 2\delta^{\alpha\delta} \text{tr}(\gamma^\beta \gamma^\epsilon) + \text{tr}(\gamma^\beta \gamma^\epsilon \gamma^\alpha \gamma^\delta) = \\ &= 8\delta^{\alpha\beta} \delta^{\delta\epsilon} - 8\delta^{\beta\delta} \delta^{\alpha\epsilon} + 2\delta^{\alpha\epsilon} \text{tr}(\gamma^\beta \gamma^\delta) - \text{tr}(\gamma^\beta \gamma^\delta \gamma^\alpha \gamma^\epsilon) \end{aligned}$$

and from cyclic property of the trace we get

$$\text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\epsilon) = 4(\delta^{\alpha\beta} \delta^{\delta\epsilon} - \delta^{\beta\delta} \delta^{\alpha\epsilon} + \delta^{\alpha\epsilon} \delta^{\beta\delta} - \delta^{\beta\epsilon} \delta^{\alpha\delta}).$$

Remember that the trace of an odd number of γ matrices vanishes: $\gamma^\mu \gamma^\mu = 4$ (sum on μ)

$$\text{tr}(\gamma^\alpha) = \frac{1}{4} \text{tr}(\gamma^\alpha \gamma^\mu \gamma^\mu) = \frac{1}{4} \delta^{\alpha\mu} \text{tr}(\gamma^\mu) - \frac{1}{4} \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\mu) \stackrel{\text{cyclicity}}{=} \frac{1}{4} \text{tr} \gamma^\alpha - \frac{1}{4} \text{tr}(\gamma^\alpha \gamma^\mu \gamma^\mu) = \left(\frac{1}{4} - 1\right) \text{tr} \gamma^\alpha$$

analogously

$$\begin{aligned} \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta) &= \frac{1}{4} \text{tr}(\gamma^\mu \gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\delta) = \frac{2}{4} \delta^{\alpha\mu} \text{tr}(\gamma^\mu \gamma^\beta \gamma^\delta) - \frac{1}{4} \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\delta) = \\ &= \frac{1}{2} \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta) - \frac{1}{2} \delta^{\beta\mu} \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\delta) + \frac{1}{4} \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\delta) = \\ &= \frac{1}{2} \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta) - \frac{1}{2} \text{tr}(\gamma^\beta \gamma^\alpha \gamma^\delta) + \frac{1}{2} \text{tr}(\gamma^\delta \gamma^\alpha \gamma^\beta) - \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta) \end{aligned}$$

$$\text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta) = -\frac{1}{2} \text{tr}(\gamma^\beta \gamma^\alpha \gamma^\delta) = -\frac{1}{2} \delta^{\alpha\beta} \text{tr}(\gamma^\delta) + \frac{1}{2} \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta) \Rightarrow \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\delta) = 0$$

Note: we did not use γ^5 since it is well defined only in $D=4$ and caution is required in dimensional regularization.

only traces of an even number of γ matrices is non vanishing, hence

$$\Pi_{\mu\nu}(p) = +e^2 \int \frac{d^D k}{(2\pi)^D} \frac{\text{tr} [\gamma_\mu (+i\not{k}) \gamma_\nu (+i\not{k} - \not{p})] + m^2 \text{tr}(\gamma_\mu \gamma_\nu)}{(k^2 + m^2)((k-p)^2 + m^2)}$$

and using the explicit expressions of the traces we find

$$\Pi_{\mu\nu}(p) = +4e^2 \int \frac{d^D k}{(2\pi)^D} \frac{-k^\mu(k-p)^\nu - k^\nu(k-p)^\mu + \delta^{\mu\nu} k(k-p) + m^2 \delta^{\mu\nu}}{(k^2+m^2)((k-p)^2+m^2)}$$

Let us combine denominators by using $\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}$ with $A = (k-p)^2+m^2$, $B = k^2+m^2$ we have

$$\begin{aligned} xA + (1-x)B &= x(k^2 - 2kp + p^2 + m^2) + (1-x)(k^2 + m^2) = \\ &= \cancel{xk^2} - 2xkp + x p^2 + \cancel{xm^2} + k^2 + m^2 - \cancel{xk^2} - \cancel{xm^2} = \\ &= k^2 - 2xkp + x p^2 + m^2 \quad \ell = k - xp, \quad \ell^2 = k^2 - 2kpx + x^2 p^2 \\ &= \ell^2 - x^2 p^2 + x p^2 + m^2 = \ell^2 + x(1-x)p^2 + m^2 = \ell^2 + \Delta \quad \text{with } \Delta = m^2 + x(1-x)p^2 \end{aligned}$$

Using $k = \ell + xp$ in the numerator of $\Pi_{\mu\nu}$ and keeping only even terms in k we find

$$\begin{aligned} \delta^{\mu\nu}(\ell+xp)(\ell-(1-x)p) - (\ell+xp)^\mu(\ell-(1-x)p)^\nu - (\ell+xp)^\nu(\ell-(1-x)p)^\mu + \delta^{\mu\nu}m^2 &= \\ = \delta^{\mu\nu}(\ell^2 - x(1-x)p^2) - 2\ell^\mu \ell^\nu + 2x(1-x)p^\mu p^\nu + \delta^{\mu\nu}m^2 + \text{linear in } k^\alpha &= \sim \ell^\mu \ell^\nu \sim \frac{1}{D} \delta^{\mu\nu} \ell^2 \\ = \delta^{\mu\nu}(1 - \frac{2}{D})\ell^2 + \delta^{\mu\nu}(m^2 - x(1-x)p^2) + 2x(1-x)p^\mu p^\nu \end{aligned}$$

Using the integral 1) of page R14 we have $\Gamma(m+1)=m!$ and $D=4-\epsilon$

$$\alpha=2, \beta=1 : \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^2}{(\ell^2+\Delta)^2} = \frac{\Delta^{(D-2)/2}}{(4\pi)^{D/2}} \frac{\Gamma(1+D/2)\Gamma(1-D/2)}{\Gamma(D/2)\Gamma(2)} = \frac{\Delta^{1-\epsilon/2}}{(4\pi)^{2-\epsilon/2}} \frac{\Gamma(3-\frac{\epsilon}{2})\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}$$

$$\alpha=2, \beta=0 : \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2+\Delta)^2} = \frac{\Delta^{(D-4)/2}}{(4\pi)^{D/2}} \frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(D/2)\Gamma(2)} = \frac{\Delta^{-\epsilon/2}}{(4\pi)^{2-\epsilon/2}} \Gamma(\epsilon/2)$$

From $\Gamma(1+z) = z\Gamma(z)$ we have $(-1+\frac{\epsilon}{2})\Gamma(-1+\frac{\epsilon}{2}) = \Gamma(\epsilon/2)$ and $1-\frac{2}{D} = 1 - \frac{1}{2-\epsilon/2} = \frac{1-\epsilon/2}{2-\epsilon/2}$

$$\left(1 - \frac{2}{D}\right) \frac{\Gamma(3-\epsilon/2)}{(-1+\epsilon/2)\Gamma(2-\epsilon/2)} = - \frac{\Gamma(3-\epsilon/2)}{(2-\frac{\epsilon}{2})\Gamma(2-\frac{\epsilon}{2})} = - \frac{\Gamma(3-\epsilon/2)}{\Gamma(3-\epsilon/2)} = -1$$

thus

$$\begin{aligned} \Pi_{\mu\nu}(p) &= 4e^2 \int_0^1 dx \frac{\Delta^{-\epsilon/2}}{(4\pi)^{2-\epsilon/2}} \Gamma(\frac{\epsilon}{2}) \left\{ \delta^{\mu\nu}(-1)\Delta + \delta^{\mu\nu}(m^2 - x(1-x)p^2) + 2x(1-x)p^\mu p^\nu \right\} \\ &= 4e^2 \int_0^1 dx \frac{\Delta^{-\epsilon/2}}{(4\pi)^{2-\epsilon/2}} \Gamma(\frac{\epsilon}{2}) \left\{ \delta^{\mu\nu}(-\cancel{m^2} - x(1-x)p^2 + \cancel{m^2} - x(1-x)p^2) + 2x(1-x)p^\mu p^\nu \right\} = \\ &= -\frac{8e^2}{(4\pi)^2} (\delta^{\mu\nu}p^2 - p^\mu p^\nu) \int_0^1 dx \Delta^{-\epsilon/2} (4\pi)^{\epsilon/2} \Gamma(\frac{\epsilon}{2}) x(1-x) \equiv \underline{(\delta^{\mu\nu}p^2 - p^\mu p^\nu) \Pi(p^2)} \quad (\text{ok with w.t.}) \end{aligned}$$

Using $\Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$ and $e \rightarrow e \bar{z}_e \mu^{\epsilon/2}$ (with $\bar{z}_e = 1$ at this order) we have

$$\begin{aligned} \Pi(p^2) &= -\frac{e^2}{2\pi^2} \int_0^1 dx \left(\frac{4\pi\mu^2}{\Delta} \right)^{\epsilon/2} \Gamma(\frac{\epsilon}{2}) x(1-x) (1+O(\epsilon^2)) = \\ &= -\frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \left(1 + \frac{\epsilon}{2} \log\left(\frac{4\pi\mu^2}{\Delta}\right) + O(\epsilon^2) \right) \left(\frac{2}{\epsilon} - \gamma + O(\epsilon) \right) (1+O(\epsilon^2)) \end{aligned}$$

thus $\Pi(p^2) = -\frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2}{\Delta}\right) - \gamma + O(\epsilon) \right)$ which introducing $\bar{\mu}^2 = 4\pi\mu^2 e^{-\gamma}$ can be rewritten in the form $\left(\int_0^1 x(1-x) = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \right)$

$$\Pi(p^2) = -\frac{e^2}{2\pi^2} \left[\frac{1}{3\epsilon} - \int_0^1 dx x(1-x) \log(\Delta/\bar{\mu}^2) \right]$$

Since $w^{(2)}(p) = m + m_1 p + \dots$ and $\tilde{\Gamma}^{(2)}(p) \tilde{W}^{(2)}(p) = 1$ we have ($e^2 = 4\pi\alpha$)

$$\begin{aligned} \tilde{\Gamma}_B^{(2)}(p) &= \delta_{\mu\nu} p^2 - (1 - \frac{1}{3}) p_\mu p_\nu - (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \Pi(p^2) = \\ &= \frac{1}{3} p_\mu p_\nu + (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \left\{ 1 + \frac{2\alpha}{3\pi\epsilon} - \frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \log\left(\frac{m^2 + x(1-x)p^2}{\bar{\mu}^2}\right) \right\} \end{aligned}$$

From $A_B = Z_A^{1/2} A$ we have $\tilde{W}_B^{(2)} = Z_A \tilde{W}^{(2)}$ and $\tilde{\Gamma}^{(2)} = Z_A \tilde{\Gamma}_B^{(2)}$ moreover $Z_S = Z_A$, thus

$$\tilde{\Gamma}^{(2)}(p) = \frac{1}{3} p_\mu p_\nu + (\delta_{\mu\nu} p^2 - p_\mu p_\nu) Z_A \left\{ 1 + \frac{2\alpha}{3\pi\epsilon} - \frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \log\left(\frac{m^2 + x(1-x)p^2}{\bar{\mu}^2}\right) \right\}$$

and in HS we have $Z_A = 1 - \frac{2\alpha}{3\pi\epsilon}$ and

$$\tilde{\Gamma}_{HS}^{(2)}(p) = \frac{1}{3} p_\mu p_\nu + (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \left\{ 1 - \frac{2\alpha_{HS}}{\pi} \int_0^1 dx x(1-x) \log\left(\frac{m^2 + x(1-x)p^2}{\bar{\mu}^2}\right) \right\}$$

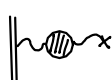
In the on-shell renormalization scheme one impose $\tilde{\Gamma}_{os}^{(2)}(p) \sim \tilde{\Gamma}_{tree}^{(2)}(p)$ for $p \simeq 0$ hence

$$Z_{A,os} = 1 - \frac{2\alpha_{os}}{3\pi\epsilon} + \frac{2\alpha_{os}}{\pi} \int_0^1 dx x(1-x) \log(m^2/\bar{\mu}^2)$$

which correspond to the substitution $\Pi(p^2) \rightarrow \Pi(p^2) - \Pi(0)$. The two point effective action is

$$\tilde{\Gamma}_{os}^{(2)}(p) = \frac{1}{3} p_\mu p_\nu + (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \left\{ 1 - \frac{2\alpha_{os}}{\pi} \int_0^1 dx x(1-x) \log\left(\frac{m^2 + x(1-x)p^2}{m^2}\right) \right\}$$

and $\alpha_{os} \simeq 1/137$ in this scheme. Note that the on-shell prescription corresponds to choosing $\bar{\mu}^2 = m^2$ in HS.

If we consider the interaction between a classical source and a classical current the one loop contribution to  gives rise to corrections to the Coulomb law known as UHELING potential, see, e.g. ITZYKSON, ZUBER "Quantum field theory" § 7-1-1, PESKIN, SCHROEDER "An introduction to quantum field theory" § 7.5, LANDAU, LIFSHITS vol. 4 § 111.

From the expression of Z_A we can evaluate the RG function $\beta(\alpha) = \mu \frac{d}{d\mu} \alpha \Big|_B$ in HS:

if we define $G = Z_e \alpha = Z_A^{-1} \alpha$ we have

$$0 = \mu \frac{d}{d\mu} \alpha \Big|_B = \mu \frac{d}{d\mu} (\mu^\epsilon Z_e \alpha) \Big|_B = \mu^\epsilon (\epsilon G + \beta(\alpha) \frac{\partial}{\partial \alpha} G) \quad \text{hence} \quad \boxed{\beta(\alpha) = -\epsilon \frac{G}{\partial G / \partial \alpha} = \frac{-\epsilon}{\frac{\partial}{\partial \alpha} \log G}}$$

Using $Z_A = 1 - \frac{2\alpha}{3\pi\epsilon}$ we have $G = \alpha/Z_A \simeq \alpha \left(1 + \frac{2\alpha}{3\pi\epsilon} \right)$ and $\log G \simeq \log \alpha + \frac{2\alpha}{3\pi\epsilon}$ thus

$$\beta(\alpha) = \frac{-\epsilon}{\frac{1}{\alpha} + \frac{2}{3\pi\epsilon}} = \frac{-\epsilon\alpha}{1 + \frac{2\alpha}{3\pi\epsilon}} \simeq -\epsilon\alpha \left(1 - \frac{2\alpha}{3\pi\epsilon} \right), \quad \text{and finally}$$

$$\beta(\alpha) = -\varepsilon\alpha + \frac{2\alpha^2}{3\pi} \quad \text{and in particular } \beta(\alpha) = \frac{2\alpha^2}{3\pi} \text{ in } D=4.$$

Using $D=4$ we have $\mu \frac{d}{d\mu} \alpha = b_0 \alpha^2$ with $b_0 = \frac{2}{3\pi} > 0$ which is analogous to the case of the ϕ^4 theory

$\frac{d\alpha}{d^2} = b_0 \frac{d\alpha}{d\mu}$, hence integrating between μ_0 and μ_1 we have $-\frac{1}{\alpha(\mu_1)} + \frac{1}{\alpha(\mu_0)} = b_0 \log \frac{\mu_1}{\mu_0}$
from which

$$\alpha(\mu_1) = \frac{\alpha(\mu_0)}{1 - b_0 \alpha(\mu_0) \log \frac{\mu_1}{\mu_0}}$$

and it is clear that the theory is infrared free since $b_0 > 0$. Note however that for $\mu_1 \rightarrow 0$ the effect of the vanishing of $\alpha(\mu_1) \sim (\log \mu_1/\mu_0)^{-1}$ is balanced in $\tilde{\Gamma}_{HS}^{(2)}(p)$ by the large log.

In the on-shell scheme, strictly speaking, there is no running coupling, however the photon two point function can be written in the form

$$\tilde{W}_{os}^{(2)}(p) = \frac{1}{\tilde{\Gamma}_{os}^{(2)}(p)} = \frac{3}{p^4} + \frac{\delta_{uv} - p_u p_v / p^2}{p^2 (1 - \Pi_{os}(p))}$$

with $\Pi_{os}(p) = \Pi(p) - \Pi(0) = \frac{2\alpha_{os}}{\pi} \int_0^1 dx \, x(1-x) \log \left(\frac{m^2 + x(1-x)p^2}{m^2} \right)$ and the interaction between two classical current is thus effectively governed by the coupling

$$\alpha_{os}(q) = \frac{\alpha_{os}}{1 - \Pi_{os}(q)}$$

NOTE: in the MS running and in $\alpha_{os}(q)$ we have considered here only the effect of electron loops, but in realistic cases also charged heavier particles have to be considered.

ABELIAN HIGGS MECHANISM

Gauge invariance prevents the possibility of adding a photon mass term to the QED Lagrangian, however it is important to note that gauge invariance does not prevent photon from acquiring a mass (by this we mean that the two point function $\langle A_\mu(x) A_\nu(y) \rangle$ can approach zero exponentially fast at large Euclidean distance or, equivalently, its Fourier transform to have in Minkowski a simple pole at $p^2 > 0$).

This fact can be explicitly seen in the so called SCHWINGER model, i.e. QED in $D=2$. If we start from the computation of the $D=4$ photon self-energy in S19-21 it is simple to get the $D=2$ result. Note that in $D=2$ there is no divergence, since the photon self-energy is log-divergent by power counting, but the degree of divergence is effectively reduced by the transversality condition.

We have, using the integral 1) of R14 and $D=2-\epsilon$

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^2}{(\ell^2 + \Delta)^2} = \frac{\Delta^{(D-2)/2}}{(4\pi)^{D/2}} \frac{\Gamma(1+D/2)\Gamma(1-D/2)}{\Gamma(D/2)\Gamma(2)} = \frac{\Delta^{-\epsilon/2}}{(4\pi)^{1-\epsilon/2}} \frac{\Gamma(2-\epsilon/2)\Gamma(\epsilon/2)}{\Gamma(1-\epsilon/2)}$$

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 + \Delta)^2} = \frac{\Delta^{(D-4)/2}}{(4\pi)^{D/2}} \frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(D/2)\Gamma(2)} = \frac{\Delta^{-1-\epsilon/2}}{(4\pi)^{1-\epsilon/2}} \Gamma(1+\epsilon/2)$$

moreover $1 - \frac{2}{D} = 1 - \frac{2}{2-\epsilon} = 1 - \frac{1}{1-\epsilon/2} = \frac{-\epsilon/2}{1-\epsilon/2}$ and using $2\Gamma(2) = \Gamma(2+1)$ we have

$$\frac{-\epsilon/2}{1-\epsilon/2} \frac{\Gamma(2-\epsilon/2)\Gamma(\epsilon/2)}{\Gamma(1-\epsilon/2)} = - \frac{\Gamma(2-\epsilon/2)\Gamma(1+\epsilon/2)}{\Gamma(2-\epsilon/2)} = -\Gamma(1+\epsilon/2)$$

The photon self energy is thus (note that there is a factor 2 of difference with respect to pag 520, since in $D=2$ fermions have two components, hence traces on spinor components give 2 instead of 4), using $\Delta = m^2 + x(1-x)p^2$

$$\begin{aligned} \Pi_{\mu\nu}(p) &= 2e^2 \int_0^1 dx \left\{ \delta^{\mu\nu} (-\Gamma(1+\epsilon/2)) \frac{\Delta^{-\epsilon/2}}{(4\pi)^{1-\epsilon/2}} + \left[\delta^{\mu\nu} (m^2 - x(1-x)p^2) + \right. \right. \\ &\quad \left. \left. + 2x(1-x)p^\mu p^\nu \right] \frac{\Delta^{-1-\epsilon/2}}{(4\pi)^{1-\epsilon/2}} \Gamma(1+\epsilon/2) \right\} = \\ &= \frac{2e^2 \Gamma(1+\epsilon/2)}{(4\pi)^{1-\epsilon/2}} \int_0^1 dx \Delta^{-1-\epsilon/2} \left\{ -\delta^{\mu\nu} (m^2 + x(1-x)p^2) + \delta^{\mu\nu} (m^2 - x(1-x)p^2) + 2x(1-x)p^\mu p^\nu \right\} = \\ &= -\frac{4e^2 \Gamma(1+\epsilon/2)}{(4\pi)^{1-\epsilon/2}} (p^2 \delta^{\mu\nu} - p^\mu p^\nu) \int_0^1 dx x(1-x) \Delta^{-1-\epsilon/2} \end{aligned}$$

We can now use $\epsilon \rightarrow 0$ and we get $\Pi_{\mu\nu}(p) = (p^2 \delta^{\mu\nu} - p^\mu p^\nu) \Pi(p)$ with

$$\Pi(p) = -\frac{e^2}{\pi} \int_0^1 dx \frac{x(1-x)}{m^2 + p^2(1-x)x}$$

If $m \neq 0$ we have that $\Pi(p)$ is regular at $p=0$ and the

photon is massless, but if $m=0$ (massless electron) we have $\Pi(p) = -\frac{e^2}{\pi} \frac{1}{p^2}$, thus

$$\tilde{\Gamma}^{(2)}(p) = \delta_{\mu\nu} p^2 - (1 - \frac{1}{\epsilon}) p_\mu p_\nu - \Pi_{\mu\nu}(p) = \frac{1}{\epsilon} p_\mu p_\nu + (\delta_{\mu\nu} p^2 - p_\mu p_\nu) (1 + \frac{e^2}{\pi} \frac{1}{p^2})$$

and

$$\tilde{W}^{(2)}(p) = \frac{\epsilon p_\mu p_\nu}{p^4} + \frac{\delta_{\mu\nu} p^2 - p_\mu p_\nu / p^2}{p^2 (1 - \Pi(p))} = \frac{\epsilon p_\mu p_\nu}{p^4} + \frac{\delta_{\mu\nu} p^2 - p_\mu p_\nu / p^2}{p^2 + \frac{e^2}{\pi}}$$

which going in Minkowski signature has (apart from the trivial ϵ -dep. pole irrelevant for matrix elements of conserved currents) a pole for $p^2 = \frac{e^2}{\pi}$, thus $M_\gamma^2 = \frac{e^2}{\pi}$. (Schwinger 1963)

(Anderson 1962 noted that the same mechanism works in the BCS theory of superconductivity)

Before considering the gauge case let us look at the single component complex scalar case

with Euclidean Lagrangian $\mathcal{L}_E = |\partial_\mu \phi|^2 + \lambda(|\phi|^2 - v^2)^2$, which is invariant for $\phi(x) \rightarrow e^{i\alpha} \phi(x)$

(note that the potential $\lambda(x^2 - v^2)^2$ corresponds to $x^2 + \lambda x^4$ with $x < 0$, indeed

$$\lambda(x^2 - v^2)^2 = \lambda x^4 - 2\lambda v^2 x^2 \quad \text{hence } v^2 = \frac{-x}{2\lambda}$$

Using the field redefinition $\phi = v + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, with $\phi_j \in \mathbb{R}$, we have

$$|\partial_\mu \phi|^2 = \left| \frac{1}{\sqrt{2}} \partial_\mu \phi_1 + i \frac{1}{\sqrt{2}} \partial_\mu \phi_2 \right|^2 = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2$$

$$|\phi|^2 = \left(v + \frac{1}{\sqrt{2}} \phi_1 \right)^2 + \frac{1}{2} \phi_2^2 = \frac{1}{2} \phi_1^2 + \frac{1}{2} \phi_2^2 + \sqrt{2} v \phi_1 + v^2$$

$$(|\phi|^2 - v^2)^2 = \left(\frac{1}{2} \phi_1^2 + \frac{1}{2} \phi_2^2 + \sqrt{2} v \phi_1 \right)^2 = \frac{1}{4} \phi_1^4 + \frac{1}{4} \phi_2^4 + 2 v^2 \phi_1^2 + \sqrt{2} v \phi_1^3 + \sqrt{2} v \phi_1 \phi_2^2 + \frac{1}{2} \phi_1^2 \phi_2^2$$

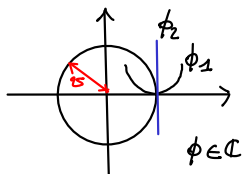
hence we can rewrite $\mathcal{L}_E$ in the form

$$\mathcal{L}_E = \frac{1}{2} (\partial_\mu \phi_1)^2 + 2\lambda v^2 \phi_1^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + \text{interactions}$$

and we see that if $v > 0$ the field ϕ_2 is massless (the Goldstone boson of $U(1) \sim SO(2) \rightarrow \{1\}$)

while ϕ_1 has a mass $m_1^2 = 4\lambda v^2$. The interpretation of this result is simple by

looking at the geometric form of the (classical) potential:



ϕ_2 is massless since it is tangent to the minimum of the potential,

while the radial mode ϕ_1 is massive.

Given this geometric interpretation the field redefinition

$$\phi(x) = \left(v + \frac{1}{\sqrt{2}} \eta(x) \right) \exp \left\{ i \frac{\theta(x)}{\sqrt{2} v} \right\} \quad \text{appears more appropriate}$$

$$\partial_\mu \phi = \frac{i}{\sqrt{2}} \partial_\mu \theta e^{i\theta/\sqrt{2}v} + \frac{1}{\sqrt{2}} \eta \frac{i \partial_\mu \theta}{v} e^{i\theta/\sqrt{2}v} + \frac{1}{\sqrt{2}} \partial_\mu \eta e^{i\theta/\sqrt{2}v} \quad \text{hence}$$

$$|\partial_\mu \phi|^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{2} \frac{\eta}{v} \right)^2 (\partial_\mu \theta)^2 + \frac{1}{2} (\partial_\mu \eta)^2 = \frac{1}{2} (\partial_\mu \theta)^2 + \frac{1}{2} (\partial_\mu \eta)^2 + \frac{\eta}{v\sqrt{2}} (\partial_\mu \theta)^2 + \frac{\eta^2}{4v^2} (\partial_\mu \theta)^2$$

$$\lambda (|\phi|^2 - v^2)^2 = \lambda \left[\left(v + \frac{1}{\sqrt{2}} \eta \right)^2 - v^2 \right]^2 = \lambda \left(\frac{1}{2} \eta^2 + \sqrt{2} v \eta \right)^2 = 2\lambda v^2 \eta^2 + \frac{1}{4} \eta^4 + \sqrt{2} \lambda v \eta^3$$

hence $m_\theta^2 = 0$ (Goldstone boson) and $m_\eta^2 = 4\lambda v^2$ as before.

Using this parametrization we see immediately that all Goldstone boson interactions

involve derivatives, hence their low momentum behavior is peculiar (see, e.g., WEINBERG

"The quantum theory of fields" vol. II, § 19.2 or, for a particular case, ZINN-SUSTIN "Quantum field theory and critical phenomena" § 13.6)

Let us now consider the so called Abelian Higgs model, which is the gauge invariant version of the

previous model: $\mathcal{L}_E = \frac{1}{4} F_{\mu\nu}^2 + |(\partial_\mu + ie A_\mu) \phi|^2 + \lambda (|\phi|^2 - v^2)^2$ (we directly consider the $v > 0$ case)

which is invariant under $\phi \rightarrow \phi(x) e^{i\alpha(x)}$, $A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha$.

If we perform in this case the change of variable $\phi(x) = v + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$, with $\phi_j \in \mathbb{R}$

$$\text{we have } |\partial_\mu \phi|^2 = \left| \frac{1}{\sqrt{2}} \partial_\mu \phi_1 + \frac{1}{\sqrt{2}} i \partial_\mu \phi_2 + ie A_\mu v + ie A_\mu \frac{1}{\sqrt{2}} \phi_1 - e A_\mu \frac{1}{\sqrt{2}} \phi_2 \right|^2 =$$

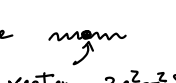
$$= \frac{1}{2} (\partial_\mu \phi_1 - e A_\mu \phi_2)^2 + \frac{1}{2} (\partial_\mu \phi_2 + e A_\mu \phi_1 + \sqrt{2} e A_\mu v)^2 = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} e^2 A_\mu^2 \phi_2^2 - e \phi_2 A_\mu \partial_\mu \phi_1 +$$

$$+ \frac{1}{2} (\partial_\mu \phi_2)^2 + \frac{1}{2} e^2 A_\mu^2 \phi_1^2 + e^2 v^2 A_\mu^2 + e \phi_1 A_\mu \partial_\mu \phi_2 + \sqrt{2} e^2 A_\mu^2 v \phi_1 + \sqrt{2} e v A_\mu \partial_\mu \phi_2$$

moreover, as in the case without gauge symmetry

$$\lambda(|\phi|^2 - v^2)^2 = 2\lambda v^2 \phi_1^2 + \text{interactions.}$$

We thus have $m_1^2 = 4\lambda v^2$ as in the case without gauge symmetry, but now we also have a term $e^2 v^2 A_\mu^2$, hence $m_2^2 = 2e^2 v^2$, however the term $\sqrt{2} e v A_\mu \partial_\mu \phi_2$ mixes A_μ and ϕ_2 so it is not immediately clear what's happening. To start clarifying this issue, and check that the Schwinger mechanism is in action here, let us consider the terms $e^2 v^2 A_\mu^2$ and $\sqrt{2} e v A_\mu \partial_\mu \phi_2$ as perturbations. Adding also the gauge fixing term $\frac{1}{2\xi} (\partial_\mu A_\mu)^2$ to the action we have:

1) photon self energy: to $O(e^2)$ the diagrams are  and 

$$\text{hence } \Pi_{\mu\nu}(p) = -2e^2 v^2 \delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} 2e^2 v^2 = -\frac{2e^2 v^2}{p^2} (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \quad \text{hence } \Pi(p^2) = -2e^2 v^2 / p^2$$

and we have for the two photon correlation function

$$W_{\mu\nu}^{(2)} = \frac{\xi p_\mu p_\nu}{p^4} + \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^2(1 - \Pi(p^2))} = \frac{\xi p_\mu p_\nu}{p^4} + \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^2 + 2e^2 v^2} \quad \text{hence } \underline{m_2^2 = 2e^2 v^2}$$

Note that the structures of $\Gamma^{(2)}$ and $W^{(2)}$ are consistent with the W-T identities. The

two point function $W_{\mu\nu}^{(2)}$ above is not simply that of a massive spin 1 particle, see later.

2) ϕ_2 self energy. Introducing the 1PI self energy $\Pi_2(p^2)$ for ϕ_2 we have for its 2-point

$$\text{function } W^{(2)}(p) = \bullet \text{---} \bullet + \bullet \text{---} \text{---} \bullet + \bullet \text{---} \text{---} \text{---} \bullet + \dots$$

$$= \frac{1}{p^2} + \frac{1}{p^2} \Pi_2(p^2) \frac{1}{p^2} + \frac{1}{p^2} \Pi_2(p^2) \frac{1}{p^2} \Pi_2(p^2) \frac{1}{p^2} + \dots = \frac{1}{p^2} \frac{1}{1 - \Pi_2(p^2)/p^2} = \frac{1}{p^2 - \Pi_2(p^2)}$$

and to leading order the only diagram contributing to $\Pi_2(p^2)$ is 

$$\text{which gives } -i p_\mu \sqrt{2} e v \left(\frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^2} + \xi \frac{p_\mu p_\nu}{p^4} \right) (-i) p_\nu \sqrt{2} e v = -2e^2 v^2 \xi$$

hence $m_2^2 = 2\xi e^2 v^2 = \xi m_2^2$. The fact that m_2^2 is gauge dependent is a strong indication that the field ϕ_2 is not a physical field. ϕ_2 is often called the would-be Goldstone boson, since in the ungauged $e \rightarrow 0$ limit it becomes the Goldstone boson.

To clarify this issue let us use the second parametrization introduced in the ungauged case

$$\phi(x) = \left[v + \frac{1}{\sqrt{2}} \eta(x) \right] \exp \left\{ \frac{i\Theta(x)}{\sqrt{2}v} \right\}$$

$$|D_\mu \phi|^2 = |(\partial_\mu + ie A_\mu) \phi|^2 = \left| \frac{1}{\sqrt{2}} \partial_\mu \eta + \frac{i}{2v} \eta \partial_\mu \Theta + ie A_\mu v + ie A_\mu \frac{1}{\sqrt{2}} \eta \right|^2$$

$$\lambda(|\phi|^2 - v^2)^2 = \lambda \left(\left(v + \frac{1}{\sqrt{2}} \eta \right)^2 - v^2 \right)^2 = \lambda \left(\frac{1}{2} \eta^2 + \sqrt{2} v \eta \right)^2 = 2\lambda v^2 \eta^2 + \eta\text{-self-int.}$$

We can now use the gauge freedom $\phi(x) \rightarrow \phi(x) e^{i\alpha(x)}$, $A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha$ to fix the unitary gauge, defined by $\Theta=0$ and $v+\eta>0$ (trivial in perturbation theory)

The Lagrangian thus takes the form

$$\begin{aligned} \mathcal{L}_E &= \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu \eta)^2 + (e v A_\mu + \frac{e}{\sqrt{2}} \eta A_\mu)^2 + 2\lambda v^2 \eta^2 + \eta\text{-self-int.} = \\ &= \frac{1}{2} (\partial_\mu \eta)^2 + 2\lambda v^2 \eta^2 + \frac{1}{4} F_{\mu\nu}^2 + e^2 v^2 A_\mu^2 + \text{interactions} \end{aligned}$$

hence $m_\eta^2 = 4\lambda v^2$, $m_\gamma^2 = 2e^2 v^2$ (as before), and the degrees of freedom corresponding to Θ, ϕ_2 have disappeared. In the unitary gauge only physical excitations exist: η (the "physical" Higgs boson) and the massive vector field.

NOTE: 1 real boson disappeared in the unitary gauge but a massive vector field has one more polarization than a massless one, so the total number of degrees of freedom is the same.

$$\begin{aligned} \int \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m^2 A_\mu^2 \right) d^4x &= \int \left(\frac{1}{2} \partial_\mu A_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} m^2 A_\mu^2 \right) d^4x = \int \frac{1}{2} A_\mu (\partial_\nu \partial_\mu - \partial_\mu \partial_\nu + m^2 \delta_{\mu\nu}) A_\nu d^4x = \\ &= \frac{1}{2} \int \tilde{A}_\mu(p) (\tilde{D}_{\mu\nu}(p)) \tilde{A}_\nu(p) d^4p \end{aligned}$$

and the propagator $\tilde{D}_{\mu\nu}(p)$ has to satisfy $(p^2 \delta_{\mu\nu} - p_\mu p_\nu + m^2 \delta_{\mu\nu}) \tilde{D}_{\nu\rho}(p) = \delta_{\mu\rho}$

Using $\tilde{D}_{\mu\nu}(p) = A \delta_{\mu\nu} + B p_\mu p_\nu$ we have

$$\delta_{\mu\rho} = (p^2 \delta_{\mu\nu} - p_\mu p_\nu + m^2 \delta_{\mu\nu}) (A \delta_{\nu\rho} + B p_\nu p_\rho) = A p^2 \delta_{\mu\rho} - A p_\mu p_\rho + A m^2 \delta_{\mu\rho} + B m^2 p_\mu p_\rho$$

thus $A(p^2 + m^2) = 1$, $-A + B m^2 = 0$ and $A = 1/(p^2 + m^2)$, $B = A/m^2$

$$\tilde{D}_{\mu\nu}(p) = \frac{\delta_{\mu\nu} + p_\mu p_\nu / m^2}{p^2 + m^2} \quad \text{photon propagator in unitary gauge (= prop. of a massive vector)}$$

This propagator is not consistent with W.T. identities: W.T. id. were derived using the gauge fixing term $\frac{1}{2\xi} (\partial_\mu A_\mu)^2$, while the unitary gauge is defined by a different gauge fixing procedure.

Note that for large momenta $\tilde{D}_{\mu\nu}(p) = O(1)$, hence power counting arguments can not be used. In particular, the renormalizability of the theory is not evident when using this gauge.

NOTE: in the Higgs mechanism there is no SSB and the vacuum is not degenerate (as signaled by the absence of Goldstone bosons). Perturbation theory manipulations are analogous to those used when SSB is present, but physics is completely different: when SSB happens different superselected sectors appear, and we can "choose" the vacuum state only by breaking the symmetry with an external field; in the gauge case we can always use local gauge transformation to rotate fields and in fact spontaneous symmetry breaking of local symmetries can be shown not to exist (ELIZUR theorem, see ITZYKSON, DROUFFE "Statistical field theory" § 6.1.3). Gauge symmetry is a redundancy of the description useful to write in a local way theories with nontrivial

constraints but it is not a physical symmetry (no multiplets!)

If we use a hard gauge fixing such that there is no local gauge invariance but only global gauge invariance (like $\vec{\nabla} \cdot \vec{A} = 0$ or $\partial_\mu A_\mu = 0$) we get something similar to the standard SSB, still some hypotheses of the Goldstone theorem fail (typically locality and manifest Lorentz invariance), see BERNSTEIN "Spontaneous symmetry breaking, gauge theories, the Higgs mechanism and all that" Rev. Mod. Phys. 46, 7 (1974) and GURALNIK, HAGEN, KIBBLE "Broken Symmetries and the Goldstone Theorem" in HARSHAK (Ed.) "Advances in Particle Physics" vol. 2 (1968).

For a gauge invariant description of the Higgs mechanism see STROCCHI "Elements of quantum mechanics of infinite systems" § C.2.5-6. ✓

Let us now introduce a gauge fixing which remove the mixing $\phi_2 \leftrightarrow A_\mu$ observed above but whose propagators vanishes for large momenta, thus allowing the use of power-counting to prove the renormalizability of the theory, the so called R_ξ gauge.

Instead of the gauge fixing $\frac{1}{2\xi}(\partial_\mu A_\mu)^2$ let us consider $\frac{1}{2\xi}(\partial_\mu A_\mu + \xi m_\gamma \phi_2)^2$, where $m_\gamma = \sqrt{2} e v$.

The Lagrangian becomes

$$\begin{aligned} \mathcal{L}_E &= \frac{1}{4} F_{\mu\nu}^2 + |\mathcal{D}_\mu \phi|^2 + \lambda (|\phi|^2 - v^2)^2 + \frac{1}{2\xi} (\partial_\mu A_\mu + \sqrt{2} e v \xi \phi_2)^2 = \\ &= \frac{1}{4} F_{\mu\nu}^2 + \left[\frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + e^2 v^2 A_\mu^2 + \sqrt{2} e v A_\mu \partial_\mu \phi_2 + \text{int} \right] + \left[2\lambda v^2 \phi_1^2 + \text{int} \right] \\ &\quad + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \xi e^2 v^2 \phi_2^2 + \sqrt{2} e v \phi_2 (\partial_\mu A_\mu) \end{aligned}$$

and the sum of the two underlined terms is a 4-divergence, irrelevant in perturbation theory.

We thus have $m_1^2 = 4\lambda v^2$, $m_\gamma^2 = 2e^2 v^2$, $m_2^2 = 2\xi e^2 v^2 = \xi m_\gamma^2$, as found before, but now there is no mixing term and we can straightforwardly compute the propagators:

$$\tilde{\Delta}_1(p) = \frac{1}{p^2 + m_1^2}, \quad \tilde{\Delta}_2(p) = \frac{1}{p^2 + m_2^2} = \frac{1}{p^2 + \xi m_\gamma^2}$$

$$\begin{aligned} \int \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m_\gamma^2 A_\mu^2 + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 \right) d^4x &= \int \frac{1}{2} A_\mu (\partial_\nu \partial_\mu - \partial_\mu^2 \delta_{\nu\mu} + m_\gamma^2 \delta_{\nu\mu} - \frac{1}{\xi} \partial_\mu \partial_\nu) A_\nu d^4x = \\ &= \frac{1}{2} \int \tilde{A}_\mu(p) \left(p^2 \delta_{\mu\nu} - p_\mu p_\nu + m_\gamma^2 \delta_{\mu\nu} + \frac{1}{\xi} p_\mu p_\nu \right) \tilde{A}_\nu(p) d^4p \end{aligned}$$

hence assuming $\tilde{\mathcal{D}}_{\mu\nu}(p) = A \delta_{\mu\nu} + B p_\mu p_\nu$ we have for the photon propagator $\tilde{\mathcal{D}}_{\mu\nu}(p)$

$$\begin{aligned} \tilde{\mathcal{D}}_{\mu\rho} &= (p^2 \delta_{\mu\nu} - p_\mu p_\nu + m_\gamma^2 \delta_{\mu\nu} + \frac{1}{\xi} p_\mu p_\nu) (A \delta_{\nu\rho} + B p_\nu p_\rho) = A p^2 \delta_{\mu\rho} - A p_\mu p_\rho + A m_\gamma^2 \delta_{\mu\rho} + A \frac{1}{\xi} p_\mu p_\rho \\ &\quad + B m_\gamma^2 p_\mu p_\rho + \frac{1}{\xi} B p^2 p_\mu p_\rho \end{aligned}$$

hence $1 = A(p^2 + m_\gamma^2)$ and $0 = A(\frac{1}{\xi} - 1) + B(m_\gamma^2 + \frac{1}{\xi} p^2)$, thus $A = \frac{1}{p^2 + m_\gamma^2}$ and

$$B = \frac{1 - 1/\xi}{p^2 + m_\gamma^2} \cdot \frac{1}{m_\gamma^2 + \frac{1}{\xi} p^2} = \frac{\xi - 1}{p^2 + m_\gamma^2} \cdot \frac{1}{p^2 + \xi m_\gamma^2}$$

and the photon propagator in R_ξ gauge is thus

$$\begin{aligned}\tilde{D}_{\mu\nu}(p) &= \frac{1}{p^2 + m_\gamma^2} \left(\delta_{\mu\nu} + (\xi - 1) \frac{p_\mu p_\nu}{p^2 + \xi m_\gamma^2} \right) = \frac{\delta_{\mu\nu} + p_\mu p_\nu / m_\gamma^2}{p^2 + m_\gamma^2} + \frac{p_\mu p_\nu}{p^2 + m_\gamma^2} \left(-\frac{1}{m_\gamma^2} + (\xi - 1) \frac{1}{p^2 + \xi m_\gamma^2} \right) = \\ &= \frac{\delta_{\mu\nu} + p_\mu p_\nu / m_\gamma^2}{p^2 + m_\gamma^2} + \frac{p_\mu p_\nu}{p^2 + m_\gamma^2} \frac{-p^2 - \xi m_\gamma^2 + \xi m_\gamma^2 - m_\gamma^2}{m_\gamma^2 (p^2 + \xi m_\gamma^2)} = \frac{\delta_{\mu\nu} + p_\mu p_\nu / m_\gamma^2}{p^2 + m_\gamma^2} - \frac{p_\mu p_\nu / m_\gamma^2}{p^2 + \xi m_\gamma^2}\end{aligned}$$

which is not the simple propagator of a massive vector particle: a further term is present, to ensure that for large p^2 we have $\tilde{D}_{\mu\nu}(p) = \mathcal{O}(p^{-2})$ and we can use power counting.

NOTE: the field ϕ_2 can be removed from the theory, thus does not correspond to an asymptotic state and to external lines, we only have to use it in internal lines. In Hinkowski signature spurious (i.e. unphysical) singularities at $p^2 = \xi m_\gamma^2$ seem to exist both in $\tilde{\Delta}_2(p)$ and in $\tilde{D}_{\mu\nu}(p)$, but they cancel in physical amplitudes.

NOTE: in QED we could use the gauge fixing term $\frac{1}{2\xi}(\partial_\mu A_\mu)^2$ since $\partial_\mu A_\mu$ is linear in the gauge transformation: $\partial_\mu A_\mu \rightarrow \partial_\mu A_\mu - \frac{1}{e} \partial_\mu^2 \alpha$. The same is no more true for $\frac{1}{2\xi}(\partial_\mu A_\mu + \xi m_\gamma \phi_2)^2$ since the gauge transformation of ϕ_2 is nontrivial ($\phi(x) \rightarrow e^{i\alpha(x)} \phi(x)$). For this reason further terms have to be added to the Lagrangian, following the FADDEEV-POPOV prescription.

The Abelian Higgs model is nothing but a relativistic version of the LANDAU-GINZBURG theory of superconductivity, whose free energy is given by

$$F = \int \left\{ \frac{B^2}{8\pi} + \frac{\hbar^2}{2m_*} |(\vec{\nabla} - \frac{ie_* \hbar c}{\hbar c} \vec{A})\psi|^2 + a|\psi(x)|^2 + \frac{b}{2} |\psi(x)|^4 \right\} d^3x$$

where $m_* = 2$ effective electron mass and $e_* = 2e$ (COOPER pairs) and $\psi(x)$ is the wavefunction of the condensed Cooper pairs.

The length $\delta = \sqrt{\frac{m_* c^2 \hbar}{4\pi e_*^2 |a|}}$ is the penetration length, and corresponds to the inverse of the photon mass. The MEISSNER effect (screening of external magnetic fields in superconductors) is thus just the analogous in this case of the Higgs mechanism.

For a discussion of the Ginzburg-London theory as an effective theory of superconductivity based on the Higgs mechanism see WEINBERG "The quantum theory of fields" vol. II, § 21.6. For deductions of the Ginzburg-London theory from BCS theory see LANDAU, LIFSHITS vol. II § 45, or ALTLAND, SIMONS "Condensed matter field theory" § 6.4. For a discussion of BCS theory from the renormalization group point of view see WEINBERG "The quantum theory of fields" vol. II, § 21.6 and, for an easier discussion, POLCHINSKI "Effective field theory and the Fermi surface" hep-th/9210046. Gauge invariance of the BCS solution has been discussed by ANDERSON and by NAMBU, see SCHRIEFFER "Superconductivity" § 8.5