Progress in Small-x Physics

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Plan:

Introduction

DGAP description of smallx

BFKL approach

Colour dipole model

Evidence for saturation

Non-linear equations

Summary

Introduction

The title of my talk is extremely general. Evidently, I have not any possibility to review all small-x physics, since it is too comprehensive. So, I'll touch only those subjects which are familiar to me.

The denotation x came from deep inelastic scattering (DIS), where it was introduced by Bjorken as

$$x = \frac{Q^2}{2pq} \ , \ \ Q^2 = -q^2 \ ,$$

where p and q are the nucleon N and virtual photon γ^* momenta correspondingly. But now the notion "Small-x Physics" is used for all processes where typical virtuality Q^2 are small compared to squared c.m.s. energy s. Such definition includes as well soft hadronic interactions, which evidently can not be described in perturbative QCD. For this interaction I remind you only that hadronic cross sections at large energies are well described by (pure phenomenological) "soft Pomeron" having, to a good approximation, the linear trajectory:

$$\alpha_1(t) = 1 + \epsilon_1 + \alpha_1' t \ ,$$

with the slope

$$\alpha_1' = 0.25 Gev^{-2} \ ,$$

which is commonly adopted, and with

$$\epsilon_1 \simeq 0.08$$

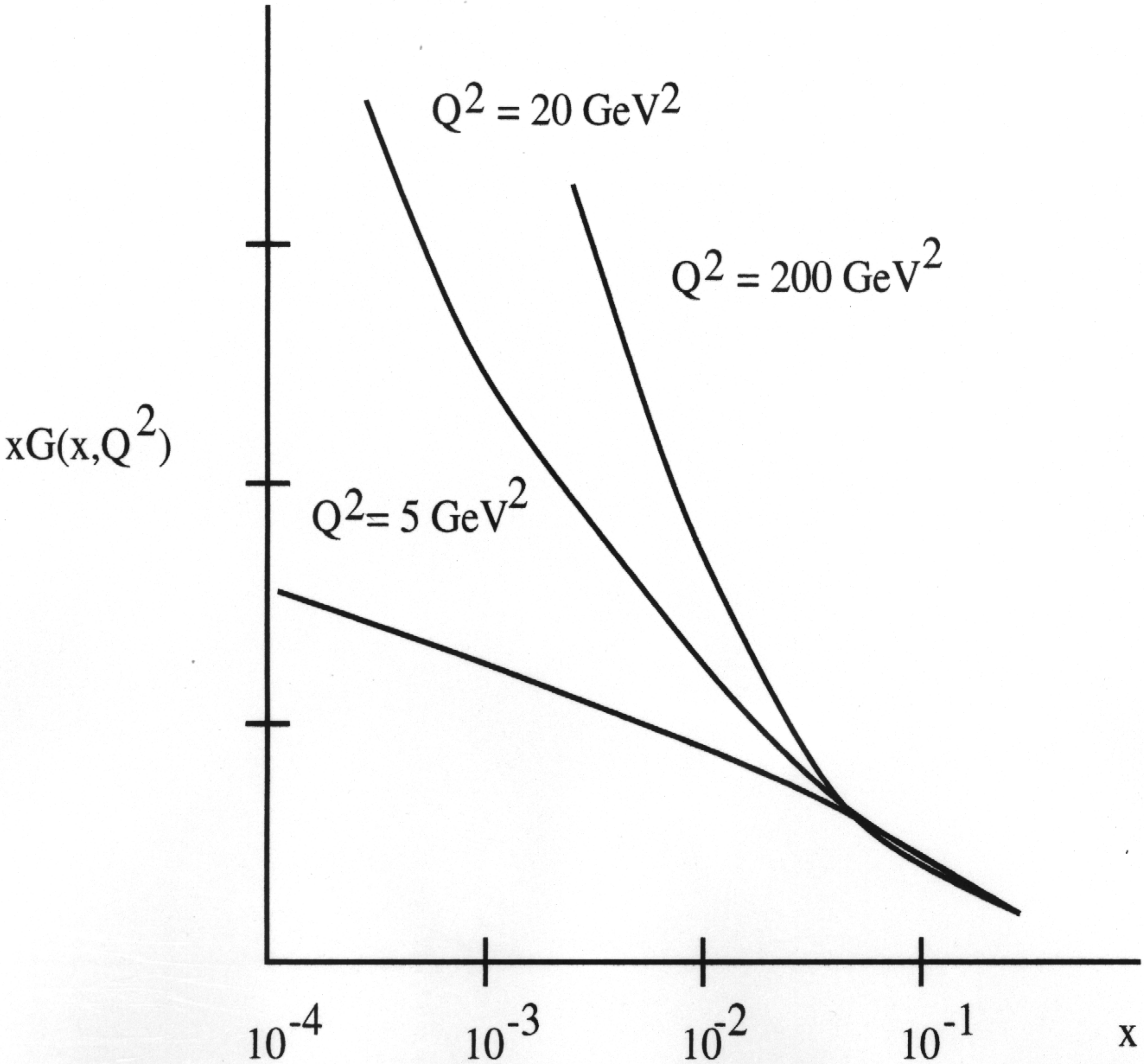
according to
A. Donnachie and P.V. Landshoff, 1992
and

$$\epsilon_1 \simeq 0.1$$

according to
F. Abe et al, CDF Collaboration, 1994;
J.R. Cudel et al, 1998;
Particle Data Group, 2000.

I'll talk about processes with "hard" scale Q^2, which justifies an application of perturbative QCD. In turn, these processes can be divided on two classes: processes with one hard scale and with two (at least) scales. The simplest representative of the first class is $\gamma^*\gamma^*$ scattering with virtualities of both photons of the same order. The total cross section of this process was measured recently at LEP. A typical example of the second kind processes is DIS, which is under investigation more than three decades. In the last decade a comprehensive and careful study of this process in the small-x region was performed at HERA. Let's start with it.

Cross sections of DIS are expressed in terms of distributions of partons (quarks and gluons) in a target (let's talk about proton, for definitness). Since at large energies $\sqrt{s}$ only gluon exchanges in t-channel give non-vanishing contributions to the cross sections, at small $x = Q^2/s$ the gluon distribution $xG(x, Q^2)$ dominates. The striking result of HERA is the strong rise of the proton structure functions with decreasing x.



x-dependence of gluon distribution for various Q^2

Standard QCD evolution (DGLAP) equation

V.V. Gribov and L.N. Lipatov, **(1972);**

L.N. Lipatov, **(1975);**

Yu.L. Dokshitzer, **(1977);**

G. Altarelli and Parisi, **(1977),**

which are basically renorm group equations, describes an evolution in Q^2. At small x the DGLAP equation may be written as

$$\frac{\partial xG(x,Q^2)}{\partial \ln Q^2} = \bar{\alpha}_S(Q^2) \int_x^1 \frac{dx'}{x'} x' G(x',Q^2) ,$$

where

$$\bar{\alpha}_S \equiv \frac{\alpha_S N_c}{\pi} ,$$

$N_c = 3$ **is the number of colours in QCD,**

$$\alpha_S(Q^2) = \frac{4\pi}{b \ln \frac{Q^2}{\Lambda^2_{QCD}}} , \quad b = \frac{11}{3} N_c - \frac{2}{3} n_f .$$

In so called infinite momentum frame (where proton is very fast) $G(x,Q^2)dx$ is a number of gluons carring fraction x of proton's momentum and transverse momenta less than Q in the interval dx. The kernel of the equation is related to the probability $dw(z,\vec{k}^{\,2})$ of the gluon splitting

$$g(1,\vec{0}) \to g(z,\vec{k}) + g(1-z,-\vec{k}) ,$$

which is

$$dw(z,\vec{k}^{\,2}) = \bar{\alpha}_S(\vec{k}^{\,2}) \frac{dz}{z} d\ln(\vec{k}^{\,2}) .$$

At small x the equation gives:

$$xG(x,Q^2) \sim \exp\sqrt{\frac{16N_c}{b} \ln\left(\frac{\ln(Q^2/\Lambda^2_{QCD})}{\ln(Q_0^2/\Lambda^2_{QCD})}\right) \ln\frac{1}{x}} ,$$

i.e. $xG(x,Q^2)$ rising with with $1/x$ (but the growth is more weak than power).

There is another evolution equation, so called BFKL equation,
V. F., E.A. Kuraev, L.N.Lipatov, (1975),
E.A. Kuraev, L.N. Lipatov, V.F., (1976),
Ya.Ya. Balitskii, L.N. Lipatov, (1978),
which describes an evolution not in $\ln Q^2$, but in $\ln(1/x)$. It predicts the power growth of cross sections with energy:

$$\sigma \sim \frac{s^{\omega_P}}{\sqrt{\ln s}} ,$$

where the Pomeron intercept (with substracted 1)

$$\omega_P = 4\bar{\alpha}_S \ln 2$$

$\simeq 0.4$ for $\alpha_s = 0.15$. Evidently this behaviour (as well as the phenomenological parametrization of hadron cross sections and the DGLAP asymptotics at small x) violate the Froissart bound $\sigma_{tot} < const(\ln s)^2$. The reasons are quite clear for both DGLAP and BFKL cases. The DGLAP equation performs resummation of terms most important in each order of perturbation theory at large $\ln Q^2$ (at fixed $\ln(1/x)$):

$(\alpha_S \ln Q^2)^n$ at leading order (LO_Q),

$\alpha_S(\alpha_S \ln Q^2)^n$ at next-to-leading order (NLO_Q)

and so on. Therefore, formally it can not be used in the region of so amall x that $\alpha_s \ln(1/x) \sim 1$ at any fixed NNN...NLO_Q order. The BFKL equation resums the terms

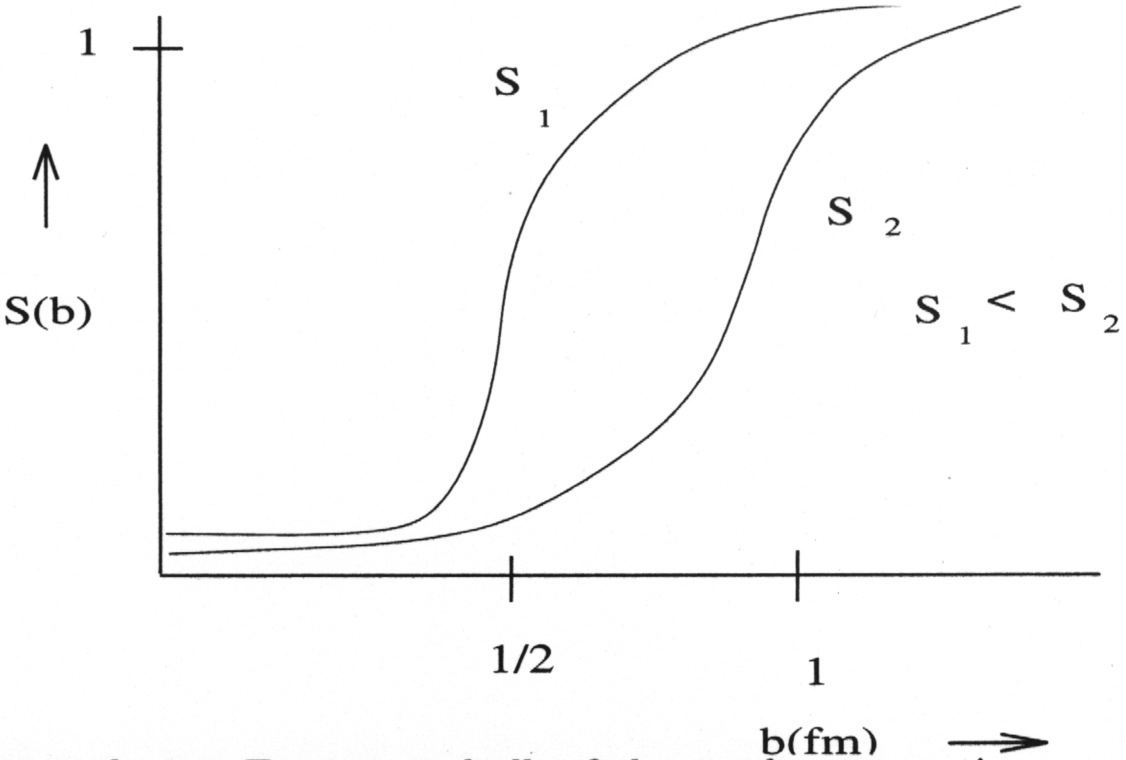
$(\alpha_S \ln(1/x))^n$ at leading order (LO_x),

$\alpha_S(\alpha_S \ln(1/x))^n$) at next-to-leading order (NLO_x).

Therefore, although it can be used in the region $\alpha_s \ln(1/x) \sim 1$, it is not applicable at asymptotically large energies (again at any fixed NNN...NLO_x order). To go to the asymptotics one need to take into account so called screening corrections to the Pomeron exchaneges, or perform unitarization.

Although the radiative corrections to the BFKL equation at any fixed order can not restore unitarity at high energy, they are very important, since in the leading order neither scale of s nor argument of the running coupling constant α_s are fixed. Therefore corrections are nesessary to fix the Pomeron intercept and normalization of cross sections. As for the unitarity, it is not clear how its restoration is important in the region of energies accessible for modern experiments.

The sucsess of the phenomenological parametrization for hadrons cross sections means that the asymptotics is not yet reached, even in hadron collisions. But of course it does not mean that corrections accounting unitarization (or screening, or shadowing corrections) are not important there. For understanding of the behaviour of the hadron cross sections an impact parameter representation for high energy scattering
U.Amaldi and G.Shubert, 1980
is very helpful. It leads to a picture for the S-matrix in the representation of the impact paranmeters b , $S(b)$,
A.H. Mueller, 1999,



which shows that at Tevatron a bulk of the total cross section comes from the region of the impact parameters where protons are almost black ($S(b)$ is small, that is unitarity limits have been reached), i.e. from the region corresponding to multiple pomeron exchange in a Regge picture
A. Capella, J. Kaplan and J. Tran Thanh Van, 1975,
A.B. Kaidalov, L.A. Ponomarev and K.A. Ter-Martirosyan, 1986.
The cross sections increase with energy due to expansion of this region. The question whether shadowing is important in processes with hard scale is not clear and will be discussed below.

First I'll say a few words about recent progress in the DGLAP approach.

NLO and NLLO DGLAP

Cross sections of processes with a hard scale Q^2 are given by convolution of parton distributions and coefficient functions (partonic cross sections). Symbolically they be may be written as

$$\mathcal{F} \otimes \hat{\sigma} \otimes \mathcal{F}$$

where $\mathcal{F}_A^i(x, Q^2)$ are distributions of partons i in hadrons A and $\sigma_{ij}(x_i, x_j, Q^2)$ are cross sections of hard interactions of the partons i and j. Evolution of the parton distributions with Q^2 is determined by the DGLAP equation, with symbolic representation

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{\bar{\alpha}_S(Q^2)}{2} \mathcal{P} \otimes \mathcal{F}$$

where $t = \ln\left(Q^2/\Lambda_{QCD^2}\right)$ and $P_j^i(z)$ are kernels of the equation, or splitting functions. Moments of the splitting functions give an anomalous dimension matrix $\gamma(N)$:

$$\gamma_{ij}(N) = \int_0^1 dz\; z^N P_j^i(z) \; .$$

Knowledge of NLO corections to the splitting functions (or, equivalently, to anomalous dimensions at arbitrary N) is necessary for accurate predictions of cross sections of hard strong-interaction processes. These corrections were calculated many years ago

G. Gurci, W. Furmanski and R. Petronzio, 1980;

E.G. Floratos, C. Kounnas and R. Lacaze, 1981

and were used since then for parton analysis. Increasing experimental accuracy calls for more precise theoretical predictions, that reguires NLLO calculations of both splitting functions and partonic cross sections.

Calculation of the NNLO corrections to the splitting functions is not yet completed, althouth a considerable progress is achived. Lowest moments of the splitting functions were recently calculated

S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, 1997;

A. Rétey and J.A.M. Vermaseren, 2001.

It gives a strong constraint for the large x behaviour. The most singular $\log 1/x$ terms are known from the BFKL kernel, which is obtained in the NLO$_x$

V.S. Fadin and L.N. Lipatov, 1998;

G. Camici and M. Ciafaloni, 1998.

Using this information, a variety of analytic expressions for the splitting functions were constructed

W.L. van Neerven and A. Vogt, 2001

with closed expressions representing the fastest and the slowest permissible evolution.

These expressions were used

A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, 2002

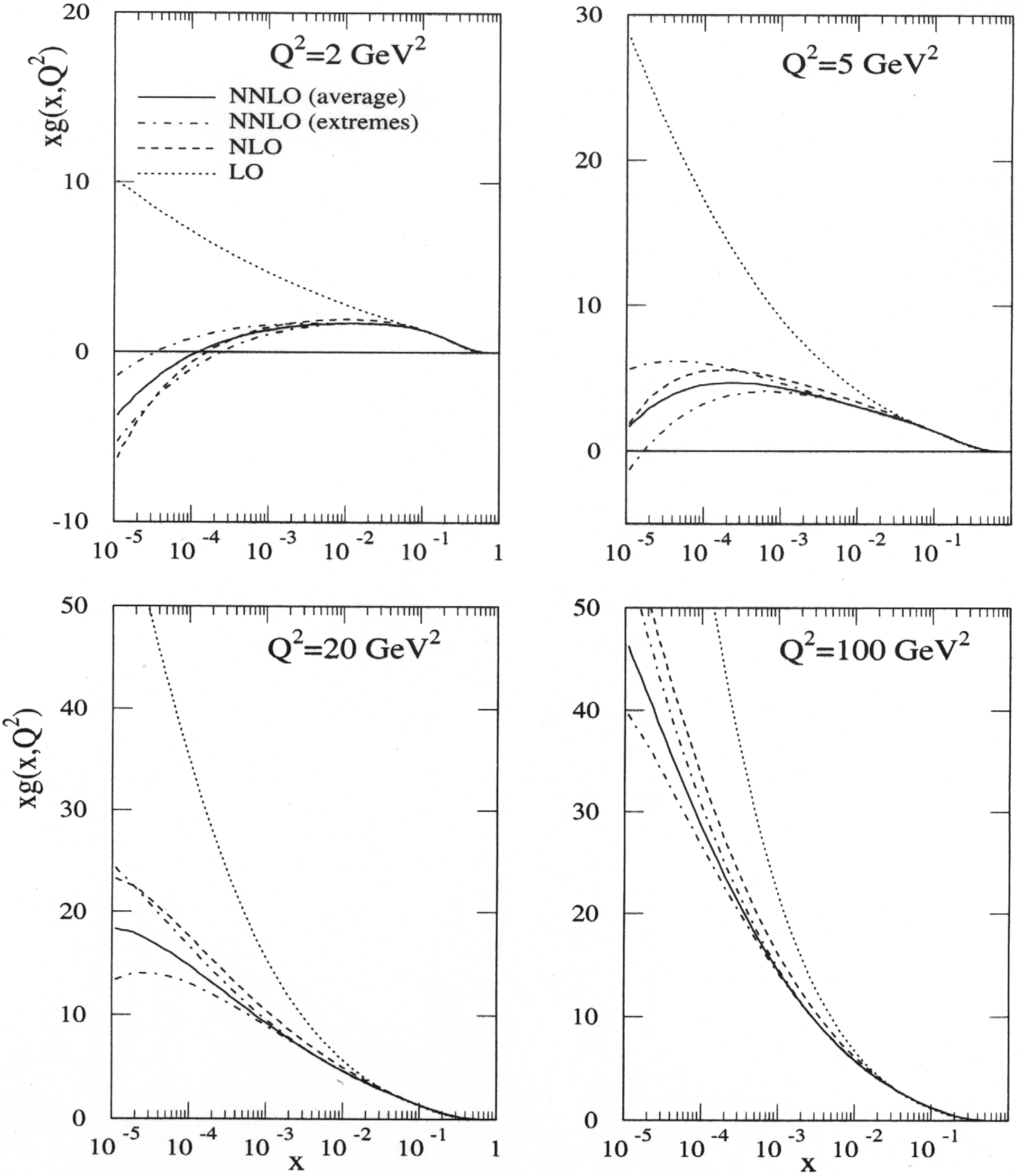
for NNLO global analysis of the data, which include the new more precise structure function measurements from HERA and the inclusive jet data from the Tevatron.

In general, compared with the NLO analysis, the NNLO one gives an improvement in quality of the descrioption of the data, although not so large as it could be expected. But the small-x behavior of the gluon distribution is intriguing.

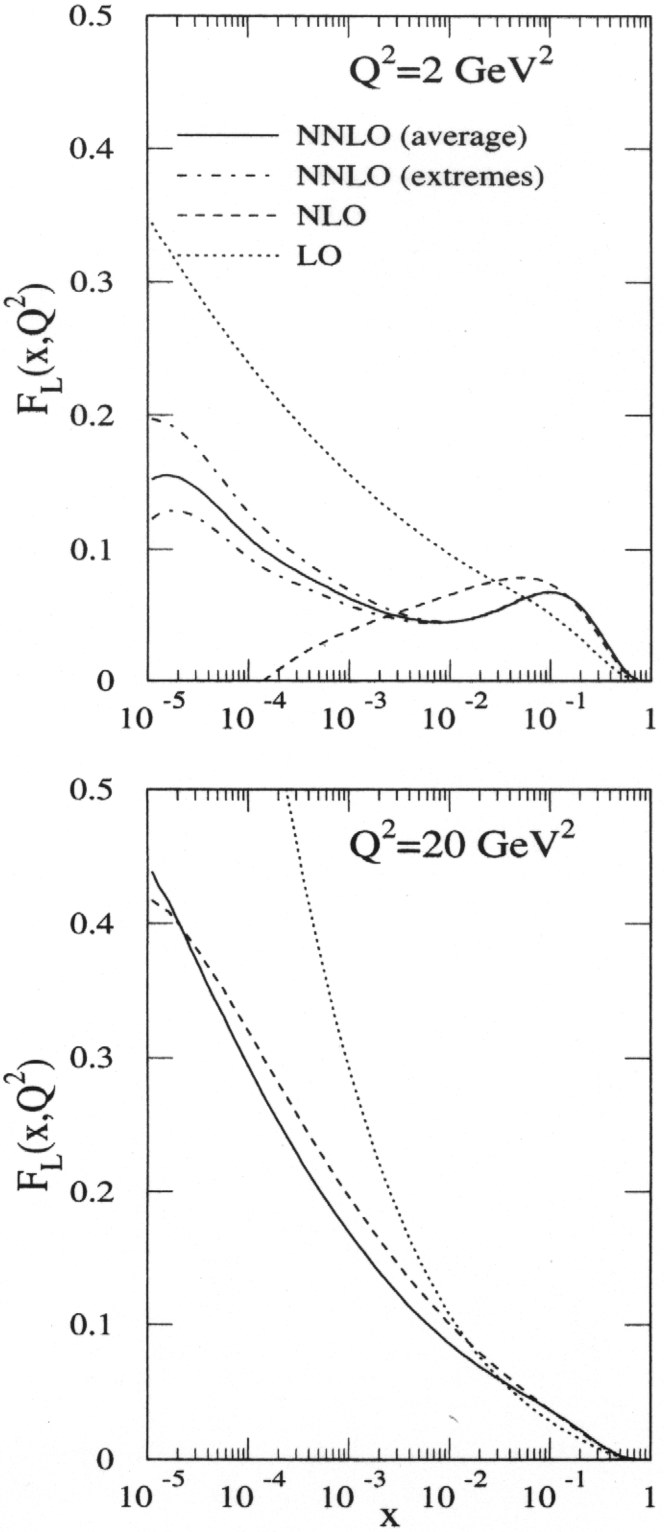
Data require gluons to be negative.

It creates a problem with F_L also.

A possible explanation is a nesessity of the small-x resummation.



Gluon distributions obtained from the LO, NLO and NNLO analyses at various values of Q^2. The three NNLO gluons result from the slow and fast extremes of the splitting functions together with the average of these.



Predictions for F_L from the LO, NLO and NNLO partons.

The partonic cross cections were known in the NNLO not long ago only for the DIS

E.B. Zijlstra and W.L. van Neerven, **1991**

and for the Drell-Yan process

R. Hamberg, W.L. van Neerven and T. Matsuura, **1991.**
E.B. Zijlstra and W.L. van Neerven, **1992.**

Recently the soft- and virtual-gluon contributions for Higgs production via gluon-gluon fusion were obtained

S. Catani, D. De Florian and M. Grazzini, **2001;**
R.V. Harlangerand W.B. Kilore, **2001.**

A remarkable progress was achieved recently in calculation of parton-parton scattering amplitudes in QCD due to analytical calculation of the basic scalar integrals for double boxes

Smirnov1,**1999;**
J.B. Tausk, **1999**

in the massless parton limit and creation of the procedure of reduction of tensor integrals to scalar ones

V.A. Smirnov and O.L. Veretin, **2000;**
C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, **2000.**

The last obstacle to the evaluation of the two-loop diagrams was overcome and a number of the two-loop amplitudes for $2 \longrightarrow 2$ scattering processes with massless partons were calculated:
Z. Bern, L. Dixon and D.A. Kosower, 2000;

Bern, L. Dixon and A. Ghinculov, 2000;
C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, 2000;
E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, 2001;
E.W.N. Glover and M.E. Tejeda-Yeomans, 2001;
S. Catani, D. De Florian and M. Grazzini, 2001.
These results were used to reobtain one of the basic parameters

in the BFKL approach - gluon Regge trajectory. It was calculated in the next-to-leading order several years ago

V. F., 1995;
V. F., R. Fiore and A. Quartarolo, 1996;
V. F., R.Fiore and M.I. Kotsky, 1996.

Recently the gluon Regge trajectory was re-evaluated
V. Del Duca, E.W.N. Glover, 2001,

in a completely independent way, by taking the high energy limit

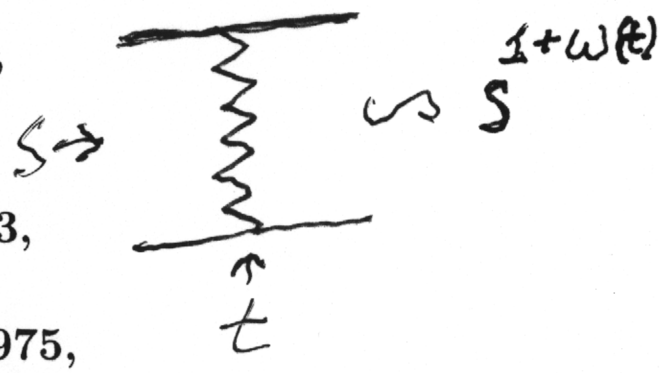
of the two-loop amplitudes for parton-parton scattering amplitudes with gluion exchanges in the t-channel. The validity of the gluon Reggeization at NLO was confirmed and full agreement with previous results was found.

At last, the trajectory of the Reggeized quark in the two-loop approximation was found by taking the high-energy limit of the two-loop amplitudes for the quark-gluon scattering
A.V. Bogdan, V. Del Duca, V. F. and E.W.N. Glover, 2002.

The quark Reggy trajectory in the LO was known from

V.F., V.E. Sherman, 1976.

BFKL (Balitskii-Fadin-Kuraev-Lipatov) approach to the description of the high energy QCD processes is based on the gluon Reggeization. In the limit of large center of mass energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$ (Regge limit) the most appropriate approach for the description of the the scattering amplitudes is given by the theory of the complex angular momenta (Gribov-Regge theory). One of the remarkable properties of QCD is the Reggeization of its elementary particles. Contrary to QED, where the electron does Reggeize in perturbation theory,
M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, F. Zachariasen, 1964,
but the photon remains elementary,
S. Mandelstam, 1964,
in QCD the gluon does Reggeize
M. T. Grisaru, H. J. Schnitzer, 1973,
L.N. Lipatov, 1976,
V. F., E.A. Kuraev, L.N.Lipatov, 1975,
E.A.Kuraev, L.N. Lipatov, V.F., 1976,
Ya.Ya. Balitskii, L.N. Lipatov,1978,
Ya.Ya.Balitskii, L.N. Lipatov, V.F., 1979,

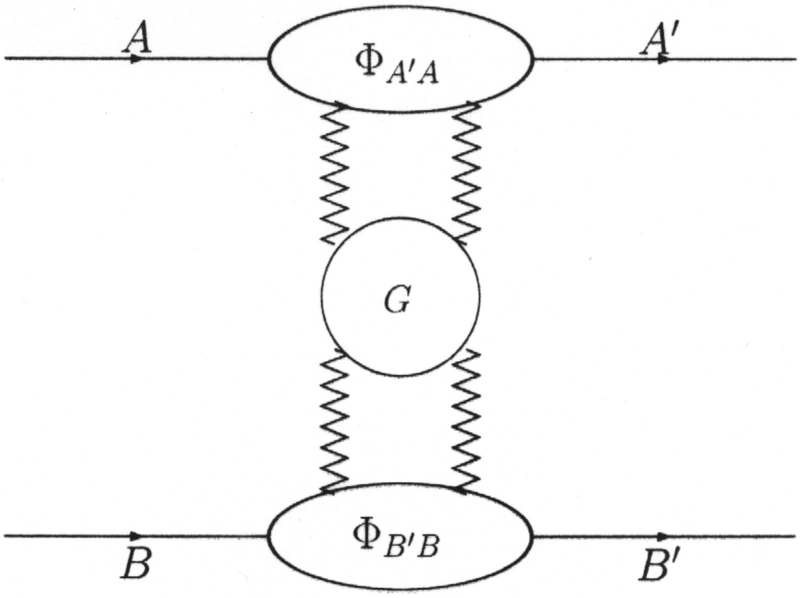


as well as the quark
V.F., V.E. Sherman, 1976.
The property of the Reggeization is very important for high energy QCD. The BFKL equation for resummation of leading logarithmic radiative correction to scatering amplitudes of processes with gluon exchanges in the t-channel is based on the gluon Reggeization. The Pomeron, determining high energy behaviour of cross sections, and the Odderon, responsible for the difference of particle and antiparticle cross sections, appears in QCD as a compound state of two and three Reggeized gluons respectively. Colorless objects constructed from Reggeized quarks and antiquarks should be relevant to phenomenological Reggeon trajectories successfully used for the description of pro-

In the BFKL approach the amplitude for the process

$$A + B \longrightarrow A' + B'$$

at large center of mass energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$, $s \gg |t|$, can be represented by the picture



and may be symbolically written as the convolution

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}$$

where the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe the transitions $A \to A'$ and $B \to B'$ due to scattering on the Reggeized gluons, while G is the Green's function for the two interacting Reggeized gluons. All dependence on properties of particles $A,\ A'$ $(B,\ B')$ is contained in the impact factors $\Phi_{A'A}$ $(\Phi_{B'B})$, which are energy independent, so that dependence on energy is determined by the universal (process independent) Green's function G. This representation is valid both in the leading logarithmic approximation (LLA), when only the leading terms $(\alpha_S \ln s)^n$ are resummed, and in the next-to-leading approximation (NLA), when the $\alpha_S(\alpha_S \ln s)^n$ terms are also resummed, not only for forward scattering with $t = 0$, but for the non-forward case as well.

In the case of the forward scattering ($A' = A, B' = B$, q=0) with the help of the optical theorem we obtain:

$$\sigma_{AB}(s) = \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \int \frac{d^2 q_A}{2\pi \vec{q}_A^{\,2}} \int \frac{d^2 q_B}{2\pi \vec{q}_B^{\,2}} \left(\frac{s}{s_0}\right)^{\omega} \Phi_A(\vec{q}_A) G_\omega(\vec{q}_A, -\vec{q}_B) \Phi_B(\vec{q}_B),$$

where the vector sign is used for vector components transverse to the initial momenta p_A, p_B and s_0 is a certain energy scale.

The Green's function obeys the equation

$$\omega G_\omega(\vec{q}_1, \vec{q}_2) = \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int d^{D-2}q_r \mathcal{K}(\vec{q}_1, \vec{q}_r)\, G_\omega(\vec{q}_r, \vec{q}_2)$$

where the Kernel

$$\mathcal{K}(\vec{q}_1, \vec{q}_2) = 2\omega\left(-\vec{q}_1^{\,2}\right)\delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \mathcal{K}_r(\vec{q}_1, \vec{q}_2)$$

is given by the sum of "virtual" part related to the gluon Regge trajectory

$$j(t) = 1 + \omega(t)\ ; \quad j(0) = 1$$

and the "real" part related to parton production in Reggeon collisions.

NLA: $\omega(t) = \omega^{(1)}(t) + \omega^{(2)}(t)\ ,\qquad \mathcal{K}_r = \mathcal{K}^{(B)}_{RRG} + \mathcal{K}^{(1)}_{RRG} + \mathcal{K}^{(B)}_{RRQ\overline{Q}} + \mathcal{K}^{(B)}_{RRGG}$

The trajectory and vertices of the Reggeized gluon were calculated in the next-to-leading order (NLO) in seriers of papers:

- $\omega^{(2)}(t)$

  [V. F. (1995)] [V. F., R. Fiore, M.I. Kotsky (1995)]
  [V. F., R. Fiore, A. Quartarolo (1996)]
  [V. F., R. Fiore, M.I. Kotsky (1996)]
  [V. F., M.I. Kotsky (1996)]

- $\gamma^{G_i\,(1\text{-loop})}_{c_i c_{i+1}}$

  [V. F., L.N. Lipatov (1993)]
  [V. F., R. Fiore, A. Quartarolo (1994)]
  [V. F., R. Fiore, M.I. Kotsky (1996)]
  [V. F., R. Fiore, A. Papa (2001)]

- $\Gamma^{c\,(1\text{-loop})}_{P'P}$

  [V. F., R. Fiore (1992)] [V. F., L.N. Lipatov (1993)]
  [V. F., R. Fiore, A. Quartarolo (1994)]
  [V. F., R. Fiore, M.I. Kotsky (1995)]

- $\gamma^{Q\overline{Q}}_{c_i c_{i+1}}$ (Born)

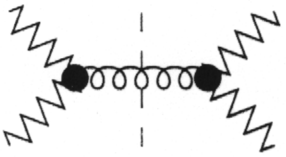
V. F., R. Fiore, A. Flachi, M.I. Kotsky (1997)]
[S. Catani, M. Ciafaloni, F. Hautmann (1990)]
[G. Camici, M. Ciafaloni (1996)]

- $\gamma^{GG}_{c_i c_{i+1}}$(Born)

[V. F., L.N. Lipatov (1996)]
[V. F., M.I. Kotsky, L.N. Lipatov (1997)]

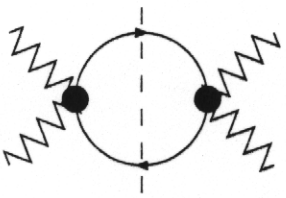
Knowledge of the Reggeon vertices gives a possibility to find the "real" part of the kernel

$$\mathcal{K}_r = \mathcal{K}^{(B)}_{RRG} + \mathcal{K}^{(1)}_{RRG} + \mathcal{K}^{(B)}_{RRQ\overline{Q}} + \mathcal{K}^{(B)}_{RRGG}$$



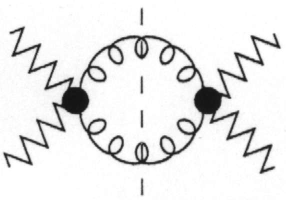
$\mathcal{K}^{(1)}_{RRG}$

[V. F., L.N. Lipatov (1993)]
[V. F., R. Fiore, A. Quartarolo (1994)]
[V. F., R. Fiore, M.I. Kotsky (1996)]
[V. F., R. Fiore, A. Papa (2001)]



$\mathcal{K}^{(B)}_{RRQ\overline{Q}}$

forward:
[V.F., R. Fiore, A. Flachi, M.I. Kotsky (1997)]
non-forward:
[V.F., R. Fiore, A. Papa, (1999)]



− counterterm

$\mathcal{K}^{(B)}_{RRGG}$

forward:
[V.F., L.N. Lipatov, M.I. Kotsky (1997)]
non-forward, octet:
[V.F., D.A. Gorbachev (2000)]

Problems of the NLO BFKL

The explicit expression for the NLO kernel obtained by

[V.F., L.N. Lipatov (1998)]
[G. Camici, M. Ciafaloni (1998)]

leads to the Pomeron intercept

$$\omega_P = \omega_P^B(1 - 2.4\,\omega_P^B) \qquad \omega_P^B = 4\ln 2N\frac{\alpha_s(\vec{q}^{\,2})}{\pi}$$

large correction!

The evident problems related to the NLO BFKL are:
— large size of the scale-invariant correction;

— running coupling.

In fact, only the first problem is really new.

The correction is so large, that for $\alpha_s = 0.15$, where $\omega_P^{(L)} \simeq 0.4$ we obtain $\omega_P \simeq 0.02$.
It means that in the HERA region the naive estimate gives the correction which almost completely cancel the main term.
the large size of the correction leads to such unpleasant phenomena as negative values for the gluon splitting function $P_{gg}(x, \alpha_s)$

R.D. Ball, S. Forte, 1998;
J. Bluemlein, V. Ravindran, W.L. van Neerven, 1998.

There was attempt to overcome the problem of the large and negative correction to the Pomeron intercept using the fact, that the corrected eigenvalue function

$$\omega(q^2,\nu) = \frac{\alpha_s(q^2)\,N_c}{\pi}\left(\chi_0(\gamma) + \frac{\alpha_s(q^2)N_c}{\pi}\chi_1(\gamma)\right)\,,\ \gamma = \frac{1}{2} + i\nu$$

completely changes its ν-dependence

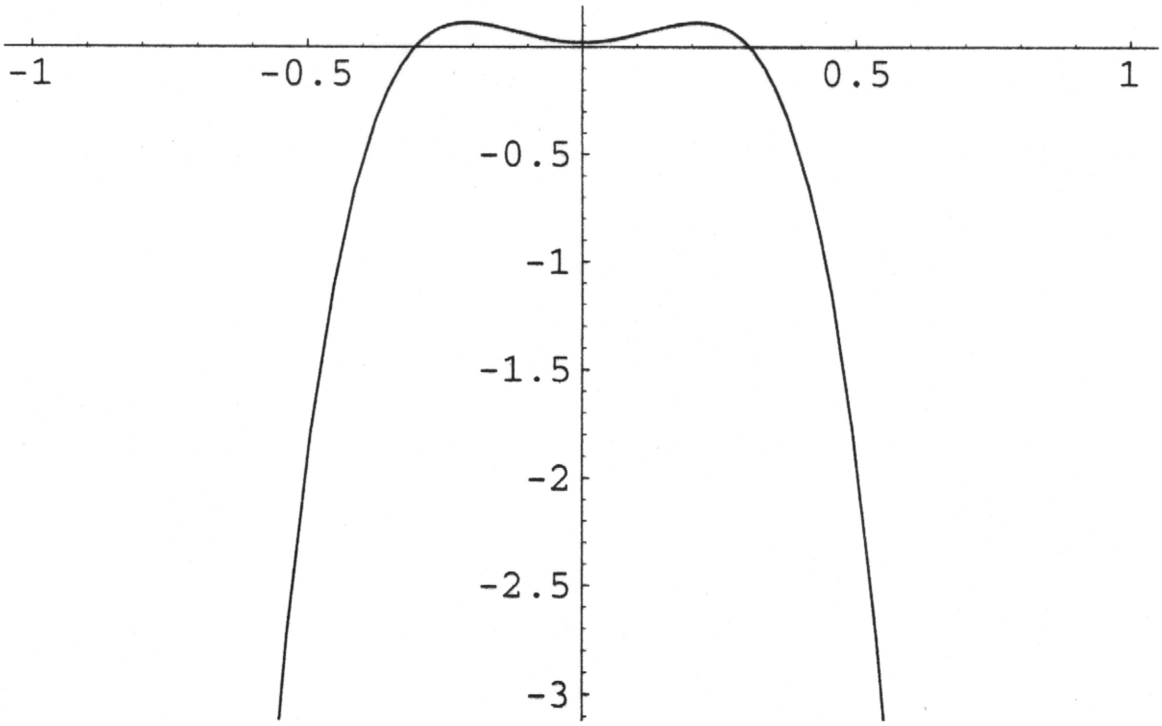
D.A. Ross, 1998.

In the LLA the point $\nu = 0$ corresponds to the maximal eigenvalue of the kernel (Pomeron intercept). In the NLO it is true only at very small α_s ($\alpha_s \leq 0.05$). For the values of α_s above 0.5 instead of having a single maximum at $\nu = 0$ the function $\omega(q^2, \nu)$ has a local minimum here. Near $\nu = 0$

$$\omega(q^2,\nu) \simeq \omega_0 + a\nu^2 - b\nu^4$$

where for $\alpha_s = 0.15$

$$\omega_0 = 0.021, \quad a = 4.19, \quad b = 47.4.$$



$\omega(\nu)$ dependence for $\alpha_s = 0.15$.

The maximum of the eigenfunction is at $\nu^2 = \frac{a}{2b}$ and

$$\sigma \sim s^{(\omega_0 + \frac{a^2}{4b})} \simeq s^{0.12}.$$

However, the solution for the Green function obtained by Ross contains oscilations.

So, it is very dezirable to diminish the relative value of the correction. It is possible to do by several ways.

—Choice of an appropriate renormalization scheme and scale setting

S.J. Brodsky, V.S. Fadin, V.T. Kim, L.N. Lipatov, G.B. Pivovarov, 1999

—Limitation from below on relative rapidities of the produced particles

C.R. Scmidt, 1999

—Improving of the equation using the renormalization group constraints

G. Salam, 1998;

M. Ciafaloni, D. Colferai, G. Salam, 1999.

Note that the last approach is developed for the applications of BFKL to DIS at small values of the Bjorken variable $x = Q^2/s$. However, the most natural region for BFKL is not DIS at small x, where two kinds of evolution are mixed: the evolution in transverse momenta of partons $\vec{k}_i^{\,2}$ from $\vec{k}_1^{\,2} \sim \Lambda_{QCD}^2$ to $\vec{k}_n^{\,2} \sim Q^2$, which is described by DGLAP, and the evolution in their longitudinal momenta fraction x_i from $x_1 \sim 1$ to $x_n =\sim Q^2/s$, which is described by BFKL. The most natural for BFKL are the processes with one (large) scale of the transverse momenta. A classic examlpe is the virtual $\gamma^*\gamma^*$ scattering.
—Choice of an appropriate renormalization scheme and scale setting

S.J. Brodsky, V.S. Fadin, V.T. Kim, L.N. Lipatov, G.B. Pivovarov, 1999

In the eigenvalue function

$$\omega(q^2, \nu) = \bar{\alpha}_s(q^2)\left(\chi_0(\gamma) + \bar{\alpha}_s^2(q^2)\chi_1(\gamma)\right),$$

where $\bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$ and $\gamma = \frac{1}{2} + i\nu$, $\chi_1(\gamma)$ is scheme and scale dependent. In the non-Abelian physical renormalization schemes (such as Υscheme) with the BLM scale setting

$$\chi_1^{BLM}(\gamma) = \chi_1(\gamma) - (\beta_0 - dependingterms)$$

the corresponding corrections are not large, that provides an opportunity for application of the NLO BFKL to high-energy phenomenology (such as $\gamma^*\gamma^*$ scattering). Moreover, the conformal invariance of the leading order is approximately conserved in this scheme;

$$\omega_P^{NLO} \simeq 0.13 \div 0.18$$

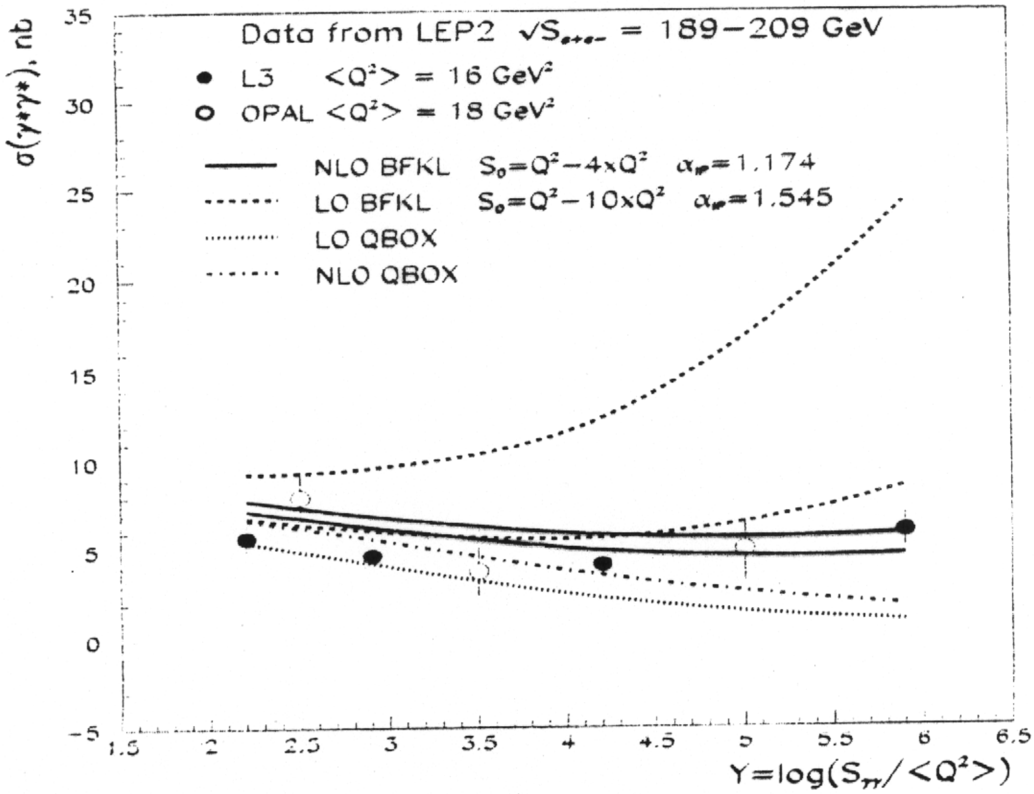


Figure 1. The energy dependence of the total cross section for highly virtual photon-photon collisions predicted by the NLO BFKL theory[11,12,6] compared to OPAL[13] and L3[14] data from LEP2 at CERN. The solid curves correspond to the BLM scale-fixed NLO BFKL predictions. The dashed curve shows the LO BFKL prediction. (Both predictions include the quark-box contribution). The BFKL predictions are shown for two different choices of the Regge scale, LO BFKL: $s_0 = Q^2 - 10Q^2$, NLO BFKL: $s_0 = Q^2 - 4Q^2$.

Another problem - running coupling. The function $\omega(q^2, \nu)$ contains a piece $-\left(\frac{N_c \alpha_s(\mu^2)}{\pi}\right)^2$ which evidently can not be considered as a small correction at all q^2. Treating formally, this piece makes questionable the BFKL approach itself:

The spectrum of the BFKL kernel in the NLA becomes unbounded, that makes the use of the complex angular momentum variable impossible

N.Armesto, J.bartels, M.A. Brown, 1998

It leads to a non-Regge behaviour of the scattering amplitudes; in particular

$$G(q_1, q_2) = \simeq \frac{1}{2\pi\sqrt{\pi D q_1^2 q_2^2 Y}} exp\left[\omega_P Y - \frac{\left(\ln q_1^2 - \ln q_2^2\right)^2}{4DY} + \frac{D}{3}\left(\alpha_s(\mu^2)\omega_P b\right)^2 Y^3\right];$$

$$Y = \ln\left(\frac{s}{s_0}\right)$$

so that the Regge-BFKL asymptotics is valid only in the limited region of energy

$$Y \leq (\alpha_s)^{-5/3}$$

Y.V. Kovchegov, A.H. Mueller, 1998;
E.Levin, 1998.

In fact, the problem of the running coupling is not new. It was clear that this problem will arise from the begining, much before the calculation of the BLO corrections. Therefore, this problem was considered by many authors:

L.N. Gribov, E.M. Levin, M.G. Ryskin, 1983;
J. Kwiechinski, 1985
L.N. Lipatov, 1986;
J.C. Collins, J. Kwiechinski, 1997;
R. E. Hancock, D.A. Ross, 1992;
L.P.A. Haakman, O. V. Kancheli, J.H. Koch, 1997;
G. Camici, M. Ciafaloni, 1997,

both for one-scale ($\gamma^*\gamma^*$) and two-scale (DIS) processes.

Whereas for the one scale processes a general solution of the problem was not found (and the results obtained here are controversal), for the two-scale processes the factorization of the Mellin transformed Green function

$$G_\omega(q_1, q_2) \simeq F_\omega(q_1)\tilde{F}_\omega(q_2)$$

at $q_1 \gg q_2$ was proved, with calculable function $F_\omega(q_1)$. It means that although the Green function itself can not be calculated in the perturbation theory, its dependence from q_1^2 is calculable. The result was confirmed after appearing of the total NLO kernel

R.S. Thorne, 1999,
M. Ciafaloni, D. Colferai, G.P. Salam, 1999.

M. Ciafaloni, M.Taiuti, A.Mueller, 2001.

When

$$\ln\frac{q_1^2}{\Lambda_{QCD}^2} \ll Y \ll \left(\ln\frac{q_1^2}{\Lambda_{QCD}^2}\right)^{\frac{5}{3}}$$

the Green function

$$G(q_1, q_2) \sim exp\left[\omega_s(t)Y\right];$$

$\omega_s(t) = \bar{\alpha}_s(t)\chi_m$, χ_m - the saddle point of $\chi(\gamma)$; $t = \ln\frac{q_1^2}{\Lambda_{QCD}^2}$, $q_1^2 \gg q_2^2$;

At $t \ll Y \ll t^2$

$$G(q_1, q_2) \sim \frac{1}{\sqrt{4\pi D\omega_s(t)Y}} exp\left[\omega_s(t)Y\left(1+\frac{\Delta t}{2t}\right) - \frac{(\Delta t)^2}{4D\omega_s(t)Y} + \frac{\eta^3}{12}\right];$$

$\Delta t = t - t_0$, $\eta = D^{\frac{1}{3}}\omega_s(t)Yt^{\frac{-2}{3}}$;

For $Y \gg t^2$

$$G(q_1, q_2) \sim exp\left[\omega_P Y\right]$$

with oscillating prefactor.

Resummation of $(\alpha_s/N_c)^n$ terms in the anomalous dimension.

T. Jaroszewicz, 1982

$$G_\omega(q_1,q_2) = \int_{C_+} \frac{d\gamma}{2\pi^2 q_2^2} \frac{exp[\gamma\left(\ln q_1^2 - \ln q_2^2\right)]}{\left(\omega - \frac{N_c\alpha_s}{\pi}\chi^B(\gamma)\right)}.$$

At $q_1^2 \gg q_2^2$:

$$G_\omega(q_1,q_2) \sim \left(q_1^2\right)^{\gamma_\omega};$$

where γ_ω is determined by the equation:

$$\omega = \frac{N_c\alpha_s}{\pi}\chi^B(\gamma_\omega).$$

It is just the renormgroup behaviour for fixed α_s. γ_ω appears as the resummed anomalous dimension.

$$\gamma_\omega = \frac{\bar{\alpha}_s}{\omega} + 2.4\left(\frac{\bar{\alpha}_s}{\omega}\right)^4 + 7\left(\frac{\bar{\alpha}_s}{\omega}\right)^6 + \ldots.$$

$$\bar{\alpha}_s = \frac{N_c\alpha_s}{\pi}$$

At NLO $\chi(\gamma)$ becomes an operator valued. Nevertheless, the BFKL equation remains fully compatible with the standard factorized perturbative evolution equations for parton distributions.

The oscillations at $Y \gg t^2$ do not affect the associated splitting functions, which remain smooth in the small x limit

[G. Altarelli, R.D. Ball, S. Forte, 2001]

Non-forward BFKL at NLA

V.F., R. Fiore, 1998

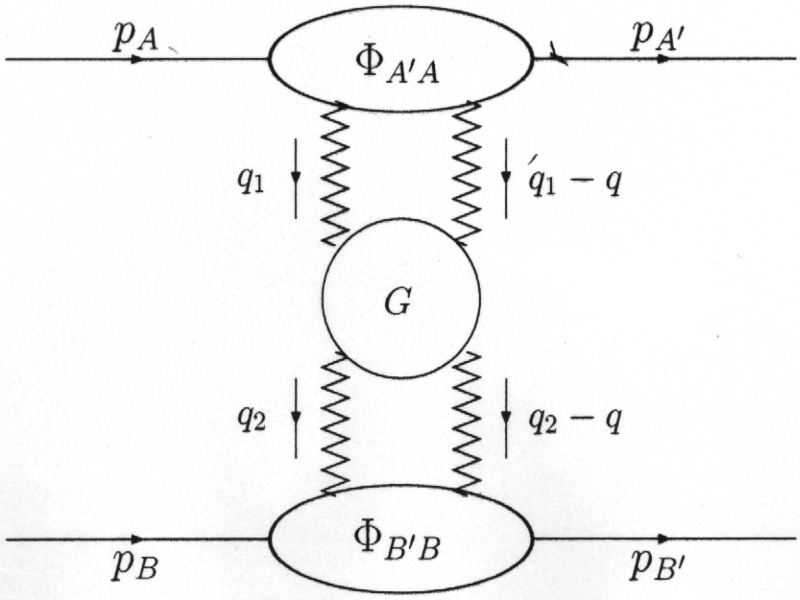
The equation discussed above (which is usually called BFKL equation) is a particular case (for forward scattering, i.e. $t = 0$ and vacuum quantum numbers in the t−channel) of the equation for the t−channel partial waves of the elastic amplitudes. We will use the term "BFKL equation" for the general case as well. The BFKL approach can predict full amplitudes of hard QCD processes in terms of the impact factors of colliding particles and of the Green's function G, which are generalizations of ones considered above. Using the

Colour decomposition:

$$\mathcal{A}_{AB}^{A'B'} = \sum_{\mathcal{R}} (\mathcal{A}_{\mathcal{R}})_{AB}^{A'B'}$$

one has for the amplitudes with definite colour representation $\mathcal{R}$) in the t-channel

$$\mathbf{Im}_s(A_{\mathcal{R}})_{AB}^{A'B'} = \frac{s}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_1}{\vec{q_1}^{\,2}(\vec{q_1}-\vec{q})^2} \int \frac{d^{D-2}q_2}{\vec{q_2}^{\,2}(\vec{q_2}-\vec{q})^2} \sum_{\nu} \Phi_{A'A}^{(\mathcal{R},\nu)}(\vec{q_1};\vec{q};s_0)$$
$$\times \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0}\right)^{\omega} G_{\omega}^{(\mathcal{R})}(\vec{q_1},\vec{q_2},\vec{q}) \right] \Phi_{B'B}^{(\mathcal{R},\nu)}(-\vec{q_2};-\vec{q};s_0)$$



$\Phi^{(\mathcal{R},\nu)}_{P'P}$ – **impact factors in the t-channel color state $(\mathcal{R},\nu)$**

$G^{(\mathcal{R})}_{\omega}$ – **Mellin transform of the Green's functions for Reggeon-Reggeon scattering in the t-channel color representation $\mathcal{R}$**

Now, The Green's functions obey the equation

$$\omega G^{(\mathcal{R})}_{\omega}(\vec{q}_1,\vec{q}_2,\vec{q}) = \vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^2\delta^{(D-2)}(\vec{q}_1-\vec{q}_2) + \int \frac{d^{D-2}q_r}{\vec{q}_r^{\,2}(\vec{q}_r-\vec{q})^2}\mathcal{K}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_r;\vec{q})\,G^{(\mathcal{R})}_{\omega}(\vec{q}_r,\vec{q}_2;\vec{q})$$

$$\mathcal{K}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2;\vec{q}) = \left[\omega\left(-\vec{q}_1^{\,2}\right)+\omega\left(-(\vec{q}_1-\vec{q})^2\right)\right]\delta^{(D-2)}(\vec{q}_1-\vec{q}_2) + \mathcal{K}^{(\mathcal{R})}_r(\vec{q}_1,\vec{q}_2;\vec{q})$$

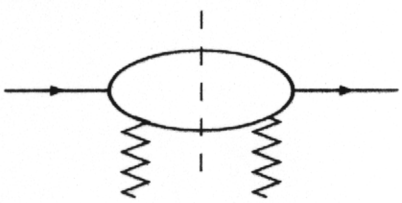
Kernel: **"virtual" part** **"real" part**

NLA: $\omega(t) = \omega^{(1)}(t) + \omega^{(2)}(t)\ ,$ $\mathcal{K}_r = \mathcal{K}^{(B)}_{RRG} + \mathcal{K}^{(1)}_{RRG} + \mathcal{K}^{(B)}_{RRQ\overline{Q}} + \mathcal{K}^{(B)}_{RRGG}$

- **The NLA kernel is known completely in the forward case ($t=0$) for the singlet color representation**
- **In the non-forward case, only the $\mathcal{K}^{(B)}_{RRGG}$ contribution is missing for the singlet color representation**
- **The NLA kernel is known completely in the non-forward case for the octet color representation**

The momentum and scale dependences of the impact factors are non-trivial.

$$\Phi_{P'P} = \sum_{\{f\}}\int\frac{d\kappa}{2\pi}d\rho_f\Gamma^c_{\{f\}A}(\Gamma^{c'}_{\{f\}A'})^*$$



– counterterm

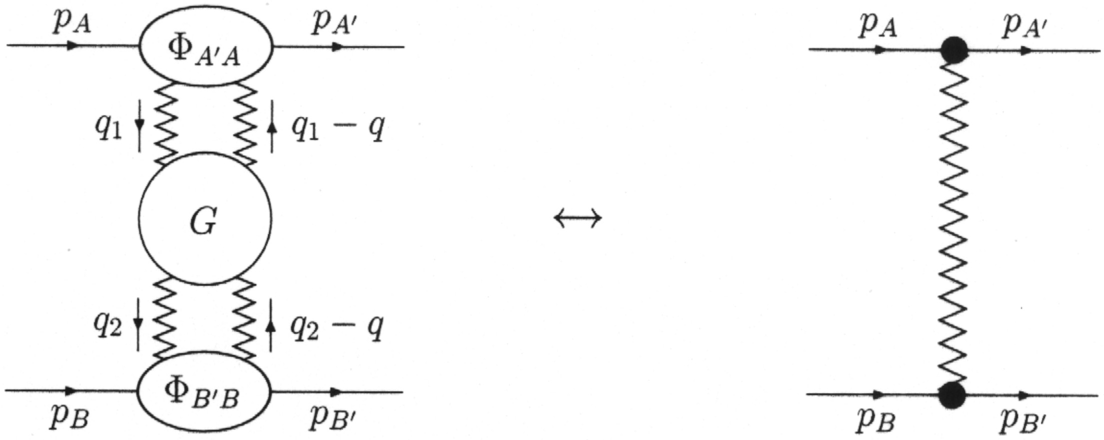
After the era of NLA corrections to the BFKL kernel (not yet over ...), the time has arrived for the calculation of impact factors in perturbative QCD.

- $\gamma^* \to \gamma^*$ **(work in progress)** **[V.F., M.I. Kotsky, D.Yu. Ivanov]** **[J. Bartels, S. Gieseke, C.F. Qiao]**
- $\gamma \to Q\overline{Q}$ **(heavy quarks)**
- ...

Bootstrap of the gluon Feggeization

Gluon quantum numbers in the t-channel

The representation for the elastic scattering process $A + B \longrightarrow A' + B'$ derived from s-channel unitarity, must reproduce the representation with one Reggeized gluon exchange in the t-channel.



In the NLA, the check of the bootstrap is very important

- **since there is no formal proof of the gluon Reggeization to all orders of perturbation theory in this approximation**
- **as a (partial) check of correctness of the NLA BFKL calculations.**

First bootstrap condition (on the NLA kernel)

$$\frac{g^2 N t}{2\,(2\pi)^{D-1}} \int \frac{d^{D-2} q_1}{\vec{q}_1^{\,2} \vec{q}_1^{\,\prime 2}} \int \frac{d^{D-2} q_2}{\vec{q}_2^{\,2} \vec{q}_2^{\,\prime 2}} \mathcal{K}^{(8)(1)} \left(\vec{q}_1, \vec{q}_2; \vec{q}\right) = \omega^{(1)}(t)\, \omega^{(2)}(t)$$

Second bootstrap condition (on the NLA impact factors)

$$\frac{ig\sqrt{N} t}{(2\pi)^{D-1}} \int \frac{d^{D-2} q_1}{\vec{q}_1^{\,2} \vec{q}_1^{\,\prime 2}} \Phi_{A'A}^{(8,a)(1)}(\vec{q}_1, \vec{q}; s_0) = \Gamma_{A'A}^{a(1)} \omega^{(1)}(t)$$
$$+ \frac{1}{2} \Gamma_{A'A}^{a(B)} \left[\omega^{(2)}(t) + \left(\omega^{(1)}(t)\right)^2 \ln\left(\frac{s_0}{\vec{q}^{\,2}}\right) \right]$$

[V.F., R. Fiore (1999)]

The first bootstrap condition has been verified

- at arbitrary space-time dimension, for the part concerning the quark contribution to the kernel (in massless QCD)
  [V.F., R. Fiore, A. Papa (1999)]
- in the $D \to 4$ limit, for the part concerning the gluon contribution to the kernel [V.F., R. Fiore, M.I. Kotsky (2000)]

The second bootstrap condition is process-dependent (it should be checked for every new impact factor which is calculated!).

It has been checked at arbitrary space-time dimension for quark and gluon impact factors in QCD with massive quarks.
[V.F., R. Fiore, M.I. Kotsky, A. Papa (2000)]

Saturation

The idea of parton saturation was developed by

L.V. Gribov, E.M. Levin, M.G. Ryskin, 1983

in the Bjorken-like frame:

$\gamma^* (0, \vec{0}, -Q)$ $\rightsquigarrow$ $\qquad$ $p\left(\frac{Q}{2x} + \frac{m^2 x}{Q}, \vec{0}, \frac{Q}{2x}\right)$ $\longleftarrow$

A sharp increase of the parton densities violates unitarity; therefore, it must be stopped due to interactions between partons in the parton cascade, and some equilibrium state of partons should be created.

Estimations:

$x\,G(x, Q^2)$ – number of gluons

$1/Q$ – transverse size

α_s^2 – probability of interaction

R — transverse size of hadron lead to a packing factor

$$\varkappa \sim \frac{\alpha_s}{Q^2 R^2} \, x\, G(x, Q^2).$$

The BFKL equation

$$\frac{\partial \rho}{\partial \ln 1/x} = \bar{\alpha}_s \, \mathcal{K} \otimes \rho \qquad (\rho = x\,G(x,Q^2))$$

at $\varkappa \gtrsim 1$ must be changed.
The gluon interactions should give in the R.H.S. a term $\sim \rho^2 \frac{\alpha_s^2}{Q^2}$ since for the interaction two partons have to meet and their interaction crossection $\sim \alpha_s^2 / Q^2$:

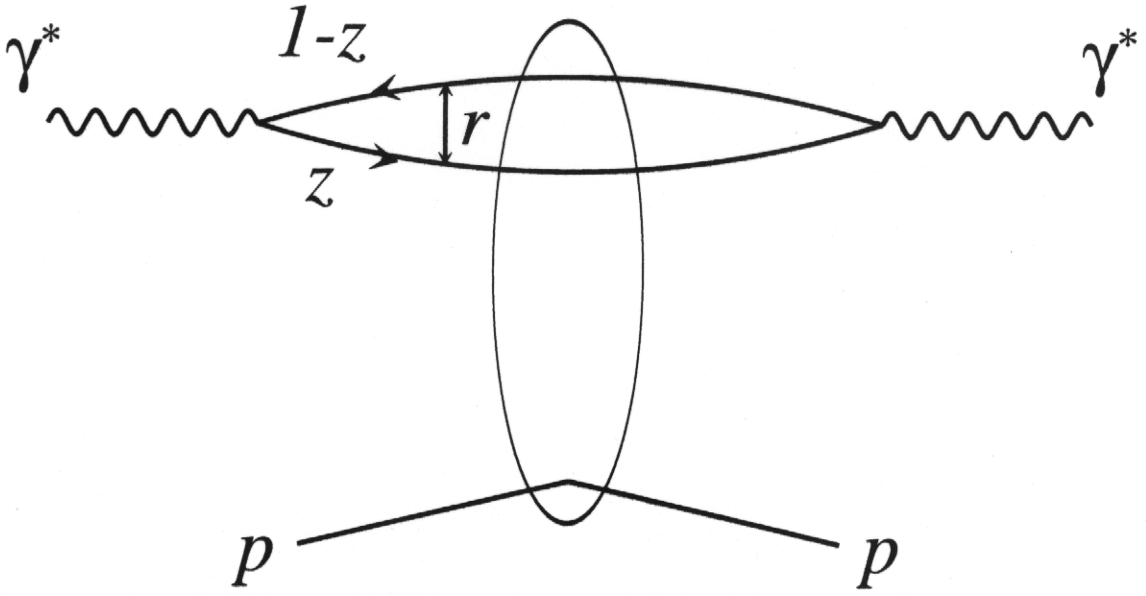
$$\frac{\partial \rho}{\partial \ln 1/x} = \bar{\alpha}_s \left(\mathcal{K} \otimes \rho - \frac{\bar{\alpha}_s}{Q^2} \, \gamma \otimes \rho \otimes \rho \right);$$

The kernel γ was not calculated.
Equation $\varkappa = 1$ determines saturation scale $Q_s^2(x)$:

$$\frac{\alpha_s(Q_s^2)}{Q_s^2 R^2} \, x\, G(x, Q_s^2) = 1.$$

Colour dipole picture

Some properties of the DIS at small x can be better understood in the laboratory frame, where the proton is at rest. In this frame the process can be represented by the picture:



Dipole picture of γ* p interaction

The virtual photon with momentum $q \simeq \frac{Q^2}{2mx}$, where m is the proton mass, decays into $q\bar{q}$ pair which lives sufficiently long time

$$\tau \sim \frac{1}{mx}$$

This time is much larger than time of interaction of fast $q\bar{q}$ pair with proton

$$t_{int} \sim R \sim \frac{1}{m} \, ,$$

where R is the proton radius.

Moreover, relative change of the distance $r_\perp \sim \frac{1}{Q}$ between quark and antiquark during the interaction time is small:

$$\frac{\Delta r_\perp}{r_\perp} \sim t_{int} \frac{k_\perp}{q} Q \sim x.$$

Thus r is a good quantum number conserved by the interaction at small x. Therefore, the $\gamma^* p$ cross section may be written as

$$\sigma_{\gamma^* p} = \int d^2\mathbf{r}\, dz\, |\Psi_{\gamma^*}(\vec{r}, z, Q^2)|^2\ \hat{\sigma}(x, r),$$

where z is the photon's momentum fraction carried by the quark, Ψ_{γ^*} is the wave function of the virtual photon and $\hat{\sigma}$ is the cross section of the interaction of the $q\bar{q}$ with the proton. This picture was developed by

N.N. Nikolaev and B.G. Zakharov, 1993;
A.H. Mueller, 1994.

It is equivalent to the BFKL approach. In the momentum space

$$\Psi_{\gamma^*}(z, \vec{k}) = \frac{e_q}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{z(1-z)}{2}}\, \frac{\bar{u}(z, \vec{k})\ \not{e}_{\gamma^*} v(1-z, -\vec{k})}{\vec{k}^{\,2} + z(1-z)Q^2}$$

and the virtual photon impact factor can be written as

$$\Phi_{\gamma^*}(\vec{q}) = \int_0^1 dz \int d\vec{r}\, |\Psi_{\gamma^*}(z, \vec{r}|^2 \frac{g^2\sqrt{N_c^2-1}}{2N_c} |1 - e^{-\vec{q}\vec{r}}|^2,$$

where

$$\Psi_{\gamma^*}(z, \vec{r}) = \int \frac{d\vec{k}}{2\pi} e^{i\vec{k}\vec{r}}\, \Psi_{\gamma^*}(z, \vec{k})\ ,$$

so that

$$|\Psi_{\gamma^*}(z, \vec{r})|^2 = \frac{6e_q^2}{\pi} \left[z^2 + (1-z)^2\right] Q^2 z(1-z)\, K_1^2(\bar{Q}r),$$

where K_1 is the Bessel function.

Saturation

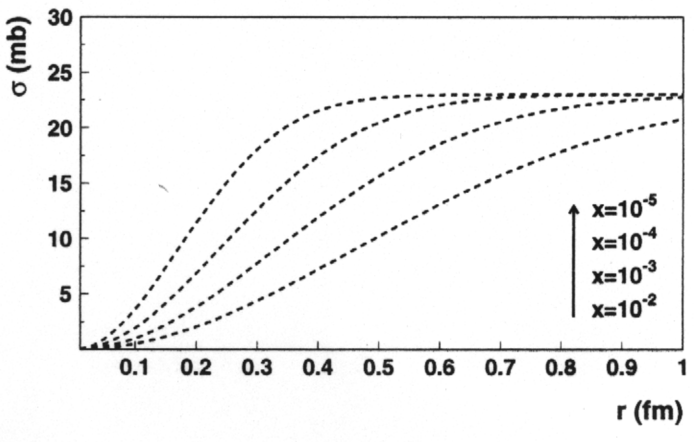
A simple model of the cross section of the interaction of the $q\bar{q}$ with the proton providing good description of inclusive and diffractive DIS at small x was suggested by
K. Golec-Biernatand M. Wusthoff, **1999.**
In this model the dipole cross section has the form

$$\hat{\sigma}(x,r) = \sigma_0 \left\{1 - \exp\left(-\frac{r^2}{4R_0^2(x)}\right)\right\},$$

where $R_0(x)$ is given by

$$R_0^2(x) = \frac{1}{Q_0^2}\left(\frac{x}{x_0}\right)^{\lambda},$$

with $Q_0^2 = 1$ GeV^2. The parameters $\sigma_0 = 23$ mb, $x_0 = 3 \cdot 10^{-4}$ and $\lambda = 0.29$ were found from the fit to all inclusive DIS data at $x < 0.01$. At small r $\hat{\sigma} \sim r^2$ (colour transparency phenomenon). For large r $\hat{\sigma} \simeq \sigma_0$ (saturation) , which can be considered as a unitarity bound. The transition between the two regimes is governed by $R_0(x)$.

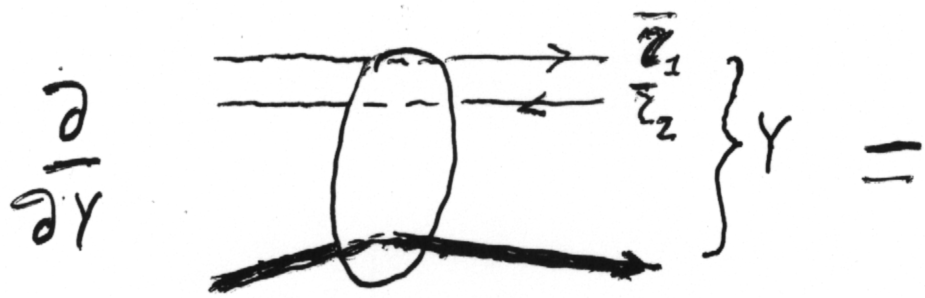


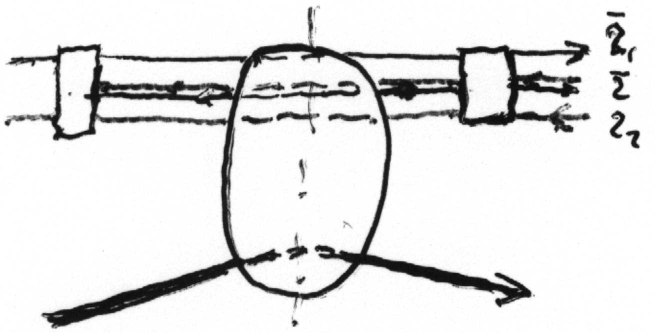
For $x \to 0$ the transition occurs for decreasing transverse sizes, in agreemnnt with analysis of
L.V. Gribov, E.M. Levin and M.G. Ryskin, **1983,**
in which the saturation scale $Q_s(x)$ was introduced.

1 The Kovchegov equation

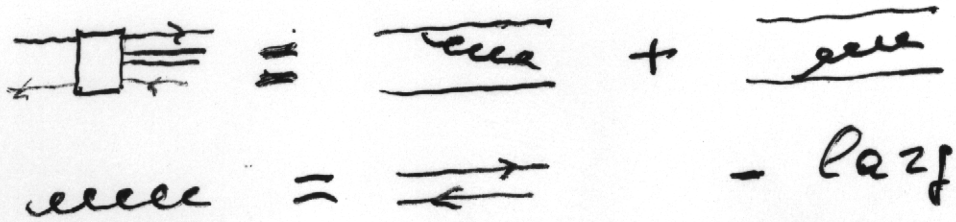
A simple equation taking into account unitarization, was derived by Yu. Kovchegov, 1999, 2000.

The derivation in the colour dipole picture looks as





— substraction



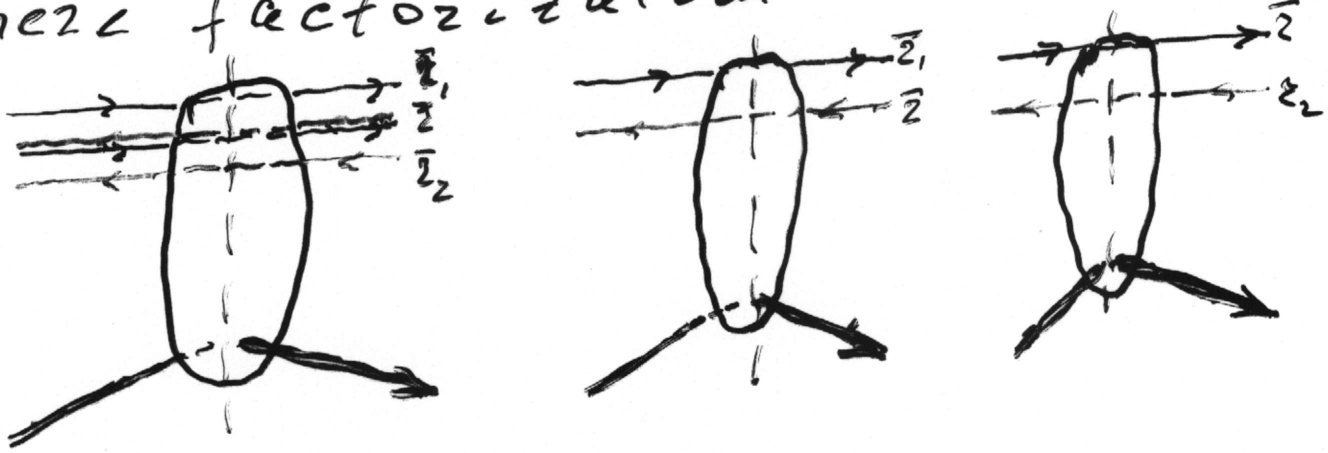
— large N_c limit

 $= \frac{\alpha_s N_c}{2\pi^2} d^2 \bar{r} \frac{(\bar{r}_1 - \bar{r}_2)^2}{(\bar{r}_1 - \bar{r})^2 (\bar{r}_2 - \bar{r})^2}$;

$$\frac{\partial S(\bar{z}_1-\bar{z}_2, Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \, \frac{(\bar{z}_1-\bar{z}_2)^2}{(\bar{z}_1-\bar{z})^2(\bar{z}_2-\bar{z})^2} \times$$

$$\times \left[S(\bar{z}_1-\bar{z}, Y)\, S(\bar{z}-\bar{z}_2, Y) - S(\bar{z}_1-\bar{z}_2, Y) \right],$$

where factorization



was assumed. It could be shown in the case of scattering on nucleus, but not obvious in the general case.

The substraction term corresponds to virtual corrections. Its coefficient is determined in the limit $S \to 1$.

For $S = 1 - iT$, $T \ll 1$

$$\frac{\partial T(\bar{z}_1-\bar{z}_2, Y)}{\partial Y} = \frac{\alpha_s N_c}{\pi^2} \int d^2 z \, \frac{(\bar{z}_1-\bar{z}_2)^2}{(\bar{z}_1-\bar{z})^2(\bar{z}_2-\bar{z})^2} \times$$

$$\times \left[T(\bar{z}_1-\bar{z}, Y) - \frac{1}{2} T(z_1-z_2, Y) \right]$$ — the dipole form of the BFKL equation

JIMWLK equation

J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, 1997, 1999; E. Iancu, A. Leonidov and L. McLerran, 2001.

The gluonic fields at high density are strong ($G_{\mu\nu} \sim \frac{1}{g}$) and semiclassical approximation can be used. Effective Lagrangian for high density QCD was built and with use of Wilson renormalization group approach the equation (presumably equivalent to Kovchegov equation) was derived.

$$\frac{dW_Y(\alpha)}{dY} = \alpha_s \left[\frac{1}{2} \int d^2x\, d^2y \, \frac{\delta^2 \; W_Y \, \eta^{ab}_{\bar{x}\bar{y}}}{\delta\alpha^a(\bar{x}, x_-)\,\delta\alpha^b(\bar{y}, y_-)} - \int d^2x \, \frac{\delta \; W_Y V^a_{\bar{x}}}{\delta\alpha^a(\bar{x}, x)} \right];$$

Summary

Considerable progress was acheived during last years in:

- DGLAP:
  moments of the spliting functions in the NNLO; two-loop parton amplitudes;

- BFKL:
  NLO forward (and partialy non-forward) kernel, impact factors, bootstrap verification;

- theory of high parton density:
  saturation of the parton densities at $x \to 0$, non-linear evolution equation, the sharp increase of the saturation scale at low x.